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MAGNETOHYDRODYNAMIC FLOW PAST A THIN AIRFOIL

E. Cumberbatch
L. Sarason
H. Weitzner

March 15, 1962

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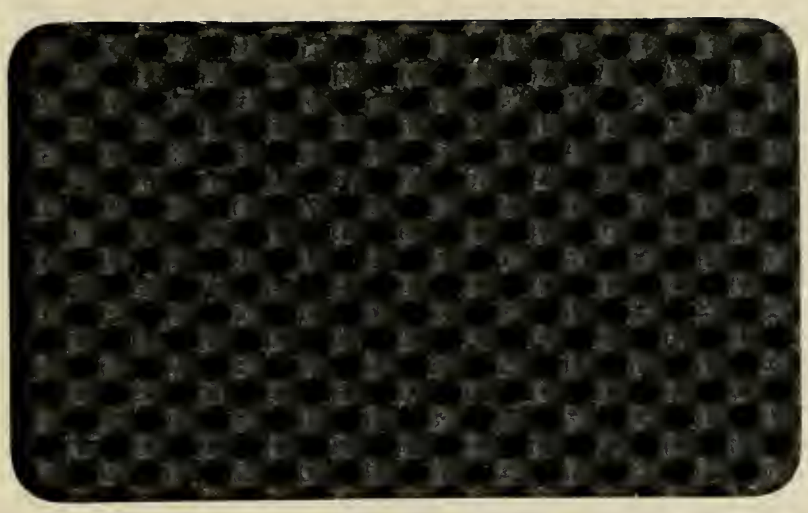

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Magneto-Fluid Dynamics Division
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Contract No. AF-49(638)-1006
Air Force Office of Scientific Research

Abstract

The steady flow of a perfectly conducting magnetohydrodynamic fluid past a thin non-conducting airfoil is studied in the model in which the fluid variables obey the Lundquist equations linearized about a constant unperturbed flow. "Hyperliptic" flows, in which hyperbolic and elliptic fields are superimposed, are considered. Results of Grad, McCune and Resler, and Sears and Resler are extended to the case of an arbitrarily inclined unperturbed field. The general solution contains four line singularities along the characteristics through the ends of the body, and has two arbitrary constants. By a "generalized Kutta-Joukowski condition" these constants are fixed so that two of the line singularities disappear. Specifically, it is required that the solution be locally square integrable. Behavior of the exponents of the singularities is investigated by numerical computation and, in limiting cases, analytically. The singular parts of some flows are investigated numerically.

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Magnetohydrodynamic Flow Past a Thin Airfoil

I. Introduction

The steady flow of a perfectly conducting magnetohydrodynamic fluid past a thin non-conducting airfoil has been studies previously.(1) The standard mathematical model chosen, and the one used here, consists of the Lundquist equations linearized about constant unperturbed fluid velocity, density and magnetic field, together with the appropriate boundary conditions.(2) Essentially two distinct types of flows appear, very closely corresponding to "subsonic" and "supersonic" flows of fluid dynamics. Since the regime similar to subsonic flow possesses properties common to both subsonic and supersonic flows the situation is more complex than either of the two fluid dynamic cases. This case is the more interesting, and with the methods outlined by Grad(2), this paper examines the "hyperliptic" flows(3) in some detail and extends the results to cases not considered previously. In much of the work presented all vectors are considered to have three components, although they are functions of two space variables only. Some sections are restricted to the cases where vectors are two dimensional and where the unperturbed flow and magnetic field have a special orientation.

In the next section the problem is formulated and Grad's results are given. In Section 3 a precise description of the hyperliptic regime is given and the formal solution to the equations is exhibited. This takes the form of a representation in terms of Cauchy integrals over the body. The boundary conditions on the body then yield a pair of coupled integral equations with a Cauchy-type kernel, whose solution is known.

The solution, however, is not unique and in Section 4 an uniqueness condition analogous to the Kutta-Joukowsky condition is proposed.(4) Similar to subsonic fluid dynamics, functions may be added which are singular at the ends of the body, by which means the singularity at either end of the body may be suppressed. The dual nature of the hyperliptic regime is evident in the fact that the singularities on the body are also propagated into the fluid along the real characteristics, and this feature is used to determine a physically acceptable uniqueness condition. The condition proposed ensures the square integrability of the flow variables over any region of the fluid. The line singularities remaining in the solution result from a breakdown of the linear analysis used and the solutions in the neighborhood of these singularities may not provide good pointwise approximations to the correct solution. On the other hand, they may give an indication of the formation of shocks.

Since the exponents of the singularities are complicated functions of the flow parameters, numerical values were obtained. A comprehensive scheme was used whereby the exponents were found over the whole hyperliptic region for a set of flow parameters, and the regions over which the trailing and leading edge singularities are to be removed could then be separated. This is described in Section 5. In the strictly two-dimensional problem a singularity is left at either the leading or trailing edge. In the limit of vanishing magnetic field this singularity is at the leading edge and hence the ordinary Kutta-Joukowsky condition is obtained. In the general problem a singularity is left at each edge. The exponent of the singularity in the solution of the restricted problem is not obtained as a uniform limit of the corresponding exponents in the solution of the general problem, for it may be a limit of either of the exponents in the general problem, depending on the choice of U_o. The flow near the edges of the body was computed for several cases of the restricted problem, and the results are given in Section 6.

II. Formulation of the Problem

The steady state equations for a perfectly conducting plasma, when linearized about a constant flow $\bar{u}_o$, magnetic field $\bar{B}_o$, density ρ_o, sound speed a_o, and magnetic permeability μ, are

$$(1a) \qquad (\bar{u}_o \cdot \bar{\nabla})\rho + \rho_o \bar{\nabla} \cdot \bar{u} = 0$$

(1b) $\rho_o(\overline{u}_o \cdot \overline{\nabla})u + a_o^2 \;\; \rho - \frac{1}{\mu}(\overline{\nabla} \times \overline{B}) \times \overline{B}_o = 0$

(1c) $(\overline{u}_o \cdot \overline{\nabla})\overline{B} - \overline{\nabla} \times (\overline{u} \times \overline{B}_o) = 0$

(1d) $\overline{\nabla} \cdot \overline{B} = 0$.

Since only flows depending on two space variables are considered, the gradient operator may be chosen as $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)$. Further, the vectors $\overline{u}_o$ and $\overline{B}_o$ are chosen as

(2a) $\overline{u}_o = u_o(1, 0, 0)$

(2b) $\overline{B}_o = B_o(\cos\theta, \sin\theta, b)$,

where u_o and B_o are the magnitudes of the projections of $\overline{u}_o$ and $\overline{B}_o$ into the xy plane. As written, the system (1) has eight equations but only seven unknowns. The system of equations actually used consists of (1) with the x-component of (1c) replaced by the condition that the perturbed flow variables vanish as $x \longrightarrow \pm \infty$ for y fixed. Under this condition the omitted equation may be derived easily from the remaining equations.

The y-component of (1c) may be integrated immediately, and with the same assymption on behavior at infinity, it follows that

(3) $B_o\cos\theta \; u_y - u_o B_y - B_o\sin\theta \; u_x = 0.$

Thus u_x may be eliminated in terms of u_y and B_y. With the introduction of non-dimensional variables defined by

(4a) $$\bar{u} = u_o(u, v, w)$$

(4b) $$\bar{B} = B_o(\xi, \eta, \zeta)$$

(4c) $$\rho = \rho_o \sigma$$

and on the definition of the Alfven speed, $A_o^2 = B_o^2/\mu\rho_o$, the basic equations become

(5a) $$\sin\theta \frac{\partial\sigma}{\partial x} + \cos\theta \frac{\partial v}{\partial x} + \sin\theta \frac{\partial v}{\partial y} - \frac{\partial\eta}{\partial x} = 0$$

(5b) $$a_o^2 \sin\theta \frac{\partial\sigma}{\partial x} + u_o^2 \cos\theta \frac{\partial v}{\partial x} + (A_o^2 \sin^2\theta - u_o^2) \frac{\partial\eta}{\partial x}$$
$$- A_o^2 \sin^2\theta \frac{\partial\xi}{\partial y} = - bA_o^2 \sin\theta \frac{\partial\zeta}{\partial x}$$

(5c) $$a_o^2 \sin\theta(\cos\theta \frac{\partial\sigma}{\partial x} + \sin\theta \frac{\partial\sigma}{\partial y}) + u_o^2 \frac{\partial v}{\partial x} - u_o^2 \cos\theta \frac{\partial\eta}{\partial x}$$
$$= - bA_o^2 \sin\theta(\cos\theta \frac{\partial\zeta}{\partial x} + \sin\theta \frac{\partial\zeta}{\partial y})$$

(5c) $$\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} = 0$$

and

$$(6a) \qquad u_o^2 \frac{\partial w}{\partial x} - A_o^2(\cos\theta \frac{\partial\zeta}{\partial x} + \sin\theta \frac{\partial\zeta}{\partial y}) = 0$$

$$(6b) \qquad \sin\theta(\cos\theta \frac{\partial w}{\partial x} + \sin\theta \frac{\partial w}{\partial y}) - \sin\theta \frac{\partial\zeta}{\partial x}$$

$$- b(\cos\theta \frac{\partial v}{\partial x} + \sin\theta \frac{\partial v}{\partial y} - \frac{\partial\eta}{\partial x}) = 0$$

(and $u \sin\theta = v \cos\theta - \eta$). If $b = 0$, the variables w and ζ disappear from (5), and the system decouples into two independent sets, (5) and (6). This strictly two-dimensional problem will be considered in more detail in some sections below. While the formulation of the problem is incomplete without boundary conditions, it is not possible to assign boundary values without a study of the solutions of the systems (5), (6) first.

As described by Grad, the system (5), (6) may readily be reduced to characteristic form. One attempts to determine a constant λ and a linear combination of the variables Z, $Z = c_1\sigma + c_2v + c_3w + c_4\xi + c_5\eta + c_6\zeta$, such that Z satisfies the equation

$$(7) \qquad (\frac{\partial}{\partial x} - \frac{1}{\lambda}\frac{\partial}{\partial y})Z = 0.$$

Such a linear combination (called a Riemann invariant) exists if and only if λ satisfies the algebraic equation

$$(8) \quad 0 = [(1+\lambda^2)(\cos\theta + \lambda\sin\theta)^2 - (1+\lambda^2)(M_o^2 + m_o^2(1+b^2)) + m_o^2M_o^2][(\lambda\sin\theta + \cos\theta)^2 - M_o^2] ,$$

where the constants m_o and M_o are defined by $m_o = u_o/a_o$, $M_o = u_o/A_o$. For the real values of λ satisfying (8), there are real linear combinations Z satisfying (7). For such real λ (7) states that Z is constant on the characteristic lines $x + \lambda y = C$. The complex roots of (8) generate complex linear combinations. In the latter case, (7) is essentially equivalent to the Cauchy-Riemann equations and states that Z is an analytic function of the complex variable $(x + \lambda y)$.

The second factor of (8) always has two real roots, and the nature of the roots of the first factor may be obtained by the simple graphical construction described by Grad (see Fig. 1). Each real tangent from the point $-\overline{u}_o$ to the characteristic cone of the time dependent system of differential equations generates the direction of the characteristic lines along which some Z is constant, and a positive direction along such lines. This direction is from $-\overline{u}_o$ to the curve. For a given characteristic, the positive direction is called "downstream", because it is the direction of the propagation of Z values for the time dependent problem. Correspondingly, the negative direction is called "upstream". The terms, "upstream" and "downstream", as defined here, need not agree with the usual definition in terms of direction of mass flow. However, the terms are relevant in that it is at the upstream ends of characteristics at which boundary conditions should be applied.

The explicit forms for the various Riemann invariants are given by Grad and are exhibited below. It is convenient to consider a vector with six components,

$$\overline{P} = \left(\frac{\sigma}{m_o^2}, v, \xi, \eta, \zeta, w\right) \tag{9}$$

and represent the invariants as

$$Z_i = \overline{r}_i \cdot \overline{P} \qquad i = 1, 2, 3, 4, 5, 6. \tag{10}$$

The two invariants associated with the quadratic factor of (8) are given by

$$r_{\pm} = \left(1, 0, \pm \frac{1}{M_o^2}, \mp \frac{\lambda_{\pm}}{M_o^2}, \frac{b}{M_o^2}, 0\right) + \left[\cos\theta \pm M_o\left(\frac{\tan^2\theta(1+\lambda_{\pm}^2)}{(1+\lambda_{\pm}^2 \tan\theta)^2} - 1\right)\right]\left(0, -\csc\theta, 0, 0 \pm \frac{1}{bM_o}, \frac{1}{b}\right) \tag{11}$$

where

$$\lambda_{\pm} = (\pm M_o - \cos\theta)\csc\theta. \tag{12}$$

The four other invariants may be expressed conveniently in terms of the parameters defined by:

(13) $$\tan \phi = \lambda$$

(14) $$\delta = \theta - \phi ,$$

so that

(15) $$\cos^2\delta_i - \cos^2\phi_i[M_o^2 + (1+b^2)m_o^2] + m_o^2M_o^2 \cos^4\phi_i = 0, \qquad i = 1, 2, 3, 4$$

and

(16) $$\overline{r}_i = (M_o^2 \sin \phi_i \cos^2\phi_i - \sin \theta \cos \delta_i ,\ \cos \phi_i (1 - M_o^2\cos^2\phi_i),$$
$$- \cos^2\phi_i \sin\delta_i ,\ \ \sin \phi_i \cos \phi_i \sin \delta_i ,$$
$$b \cos^2\phi_i \sin \phi_i ,\ \ b \cos \phi_i \sin \theta) .$$

Consistent with the linearization, the boundary conditions will be imposed on a slit, for convenience taken as $y = 0$, $|x| < 1$. The limit of a quantity taken as a point approaches the top or bottom of the airfoil is denoted by the subscript t or b The condition that the velocity normal to the body vanishes at the body, when linearized, gives v_t and v_b as known functions of x. It is also assumed that the magnetic field is continuous across the body and further, from the condition $\nabla \times B = 0$ inside the nonconducting body, $\zeta_t = \zeta_b =$ constant.

The remaining conditions depend on the nature of the solutions considered. All "eliptic" Riemann invariants are assumed to vanish at infinity and no further restriction is imposed. For the hyperbolic invariants the situation depends on the orientation of the characteristics. If the characteristic does not pass through the body, then the invariant vanishes on that line. For characteristics passing through the body, the invariant vanishes on the half-line which has infinity "upstream". On the other half-line no condition is imposed.

III. The Hyperliptic Case: General Solution

In the region of interest in this paper the characteristic equation (8) has one pair of complex conjugate roots, denoted by $\lambda_2 = s + it$ and $\lambda_3 = \overline{\lambda}_2$. The Riemann invariants Z_i, $i = 1, 4, 5, 6$ corresponding to the real characteristic roots, are zero on half lines leaving the body and with infinity upstream. Thus, by convention the boundary conditions may be taken as

(17a) $\quad (Z_i)_t = 0 \qquad i = 1, 5 \qquad \text{on } |x| < 1$

(17b) $\quad (Z_i)_b = 0 \qquad i = 4, 6 .$

For the complex roots of (8), if Z is defined by

$$(18) \qquad z = x + sy + ity = X + iY ,$$

then Z_2 is an analytic function of z, which, by the boundary conditions, vanishes at infinity. Under the transformation implied by (18), the slit is unchanged

The function Z_2 may be expressed in terms of its real and imaginary parts as

$$Z_2 = W_1 + iW_2 .$$

Since Z_2 is an analytic function of $z = X + iY$, it may be represented as a Cauchy integral in terms of its boundary values. Under the assumption that Z_2 tends to zero at infinity sufficiently rapidly, this representation has the form

$$(19) \qquad Z_2 = \frac{1}{\pi i} \int_{-1}^{1} \frac{F(\rho) + iG(\rho)}{\rho - z} \, d\rho .$$

Thus Z_2 must go to zero at infinity at least as fast as z^{-1}. The functions F and G in (19) are defined by

$$(20a) \qquad F(X) = \frac{1}{2} \left\{ (W_1)_t - (W_1)_b \right\}$$

$$(20b) \qquad G(X) = \frac{1}{2} \left\{ (W_2)_t - (W_2)_b \right\} .$$

The boundary values of W_1, W_2 are determined from (19) as

$$(W_1(X))_t = F(X) + \frac{1}{\pi}\int_{-1}^{1} \frac{G(\rho)}{\rho - X}\,d\rho \tag{21a}$$

$$(W_1(X))_b = -F(X) + \frac{1}{\pi}\int_{-1}^{1} \frac{G(\rho)}{\rho - X}\,d\rho \tag{21b}$$

$$(W_2(X))_t = G(X) - \frac{1}{\pi}\int_{-1}^{1} \frac{F(\rho)}{\rho - X}\,d\rho \tag{21c}$$

$$(W_2(X))_b = -G(X) - \frac{1}{\pi}\int_{-1}^{1} \frac{F(\rho)}{\rho - X}\,d\rho \tag{21d}$$

where the integrals are defined as Cauchy principal values. Clearly (20) and either equation for $W_1(X)$ and for $W_2(X)$ from (21) is equivalent to the system (21). Thus the fundamental equations become (17), (20) and two appropriate equations from (21).

The system of equations (17) and (20) may be considered a system of six linear equations in the eight unknowns σ_t, σ_b, ξ, η, w_t, w_b, F, and G. The inhomogeneous terms in the system are linear functions of v_t, v_b and ζ_o, functions given by the boundary conditions. Thus σ_t, σ_b, ξ, η, w_t, w_b may be expressed as a linear combination of F, G, and the known functions v_t, v_b and ζ_o. The form of the two integral equations taken from (21) becomes

$$AH + \frac{1}{\pi} B \int_{-1}^{1} \frac{H(\rho)}{\rho - X}\,d\rho = C \tag{22}$$

where A and B are two-by-two matrices with constant coefficients, and C is a two-component vector which is a known function of X (a function of v_t, v_b and ζ_o) and H represents the two-component vector (F,G). The matrix A is assumed to be nonsingular, and pre-multiplication of (22) by A^{-1} gives

$$H + \frac{1}{\pi} B' \int_{-1}^{1} \frac{H(\rho)}{\rho - X} d\rho = C' . \tag{23}$$

There exists a nonsingular matrix Q such that $\hat{B} = QB'Q^{-1}$ is in sub-diagonal form

$$\hat{B} = \begin{pmatrix} \mu & 0 \\ a & \nu \end{pmatrix} , \tag{24}$$

where $a \neq 0$ only if $\mu = \nu$. Under the transformation $\hat{H} = QH$ equation (23) becomes

$$\hat{H} + \frac{1}{\pi} \hat{B} \int_{-1}^{1} \frac{\hat{H}(\rho)}{\rho - X} d\rho = QC' , \tag{25}$$

which represents the pair of ihhomogeneous singular integral equations with a Cauchy-type kernel

$$\hat{H}_1 + \frac{\mu}{\pi} \int_{-1}^{1} \frac{\hat{H}_1(\rho)}{\rho - X} d\rho = (QC')_1 \tag{26a}$$

$$\hat{H}_2 + \frac{\nu}{\pi} \int_{-1}^{1} \frac{\hat{H}_2(\rho)}{\rho - X} d\rho = (QC')_2 - \frac{a}{\pi} \int_{-1}^{1} \frac{\hat{H}_1(\rho)}{\rho - X} d\rho . \tag{26b}$$

Provided $\mu^2, \nu^2 \neq -1$, equation (26a) is solved for $\hat{H}_1$ by standard techniques (see, for instance, Mikhlin(5)) and the solution is substituted in the right hand side of (26b) which is then solved for $\hat{H}_2$.

The solution of an integral equation of the type

$$\phi(X) + \frac{\delta}{\pi} \int_{-1}^{1} \frac{\phi(\rho)}{\rho - X} \, d\rho = f(X) \tag{27}$$

is given by

$$\begin{aligned} \phi(X) &= \frac{1}{1+\delta^2} f(X) \\ &- \frac{1}{\pi} \frac{\delta}{1+\delta^2} (1+X)^{m-1}(1-X)^{-m} \int_{-1}^{1} (1+\rho)^{1-m}(1-\rho)^m f(\rho)(\rho - X)^{-1} \, d\rho \\ &+ K(1+X)^{m-1}(1-X)^m , \end{aligned} \tag{28}$$

where m is determined from

$$e^{2\pi i m} = \frac{1 + i\delta}{1 - i\delta} \qquad 0 \leq \mathrm{Re}(m) < 1. \tag{29}$$

It is noted that the solution (28) contains the arbitrary constant K. If $f(x)$ is well-behaved, e.g., Lipshitz continuous, then it may be shown easily that

$$\underset{\substack{x \to 1 \\ x < 1}}{\phi(x)} = \frac{1}{2^{1-m}} \cdot \frac{(C+K)}{(1-x)^m} + O((1-x)^{1-m}) \tag{30a}$$

$$(30b) \qquad \underset{\substack{x \to -1 \\ x > -1}}{\phi(x)} = \frac{1}{2^m} \cdot \frac{(C'+K)}{(1+x)^{1-m}} + O((1+x)^m) \ ,$$

where

$$(31a) \qquad C = -\frac{\delta}{\pi(1+\delta^2)} \int_{-1}^{1} d\rho\,(f(\rho) - f(1)) \left(\frac{1+\rho}{1-\rho}\right)^{1-m}$$

$$(31b) \qquad C' = \frac{\delta}{\pi(1+\delta^2)} \int_{-1}^{1} d\rho\,(f(\rho) - f(-1)) \left(\frac{1-\rho}{1+\rho}\right)^{m} .$$

Hence K may be chosen so that $\phi(x)$ is regular (and even vanishes) at the leading edge or at the trailing edge. However, $\phi(x)$ will be singular at one edge or the other unless f is specially chosen. Such a choice of f implies a special shaping of the body, as is considered briefly below (p. 24).

With the solutions to the integral equations of the form (26), it remains to describe the solution of the original problem. $\hat{H}_1(x)$ and $\hat{H}_2(x)$, linear combinations of $F(x)$ and $G(x)$, satisfied (26); thus $\hat{H}_1(x)$ and $\hat{H}_2(x)$ may be written in the form (28) with constants m_1, K_1 and m_2, K_2. Further, near the ends of the body, $\hat{H}_1(x)$ and $\hat{H}_2(x)$ have the behavior given in (30) with the appropriate m and K. It follows that $F(x)$ and $G(x)$ are each the sum of two expressions of the form (28) with constants m_1, K_1 and m_2, K_2. Since $Z_2(z)$ is given by (19), it is easily seen that for z near the leading or trailing edge. i.e., $y \to 0$, $x \to \pm 1$, (32a) $$Z_2(z)_{z\to +1} = \frac{\alpha}{(1-z)^{m_1}} + \frac{\beta}{(1-z)^{m_2}} + O(1)$$

(32b) $$Z_2(z)_{z\to -1} = \frac{\alpha'}{(1+z)^{1-m_1}} + \frac{\beta'}{(1+z)^{1-m_2}} + O(1) \ .$$

Again, with judicious choice of the two constants K_1 and K_2, one constant of each pair of constants (α,α') and (β,β') may be set to zero.

Now that the elliptic part of the solution has been completed, the hyperbolic part must also be given. Since $F(x)$ and $G(x)$ are known, equations (17) and (20) constitute six equations for the six unknowns σ_t, σ_b, ξ, η, w_t, and w_b. Thus these variables are given as a linear combination of $F(x)$, $G(x)$ and the known functions $v_t(x)$, $v_b(x)$, and ζ_o. The Riemann invariants previously unspecified on the body have the form

(33a) $$(Z_i(x))_b = \alpha_i F(x) + \beta_i G(x) + R_i(x),$$
$$|x| < 1, \quad i = 1, 5 ;$$

(33b) $$(Z_i(x))_t = \alpha_i F(x) + \beta_i G(x) + R_i(x),$$
$$|x| < 1, \quad i = 4, 6 ,$$

where $R_i(x)$ are well-behaved functions of x. Since the i^{th}

Riemann invariant is constant on the line $x + y\lambda_i$, the non-zero values of the Riemann invariants are given everywhere by (33) on the replacement $x \rightarrow x + y\lambda_i$. Hence the singularities present in F(x) and G(x) at the tips of the body propagate along characteristics. In general, the real Riemann invariants behave like

$$\text{(34a)} \qquad \underset{\substack{x+y\lambda_i \to 1 \\ x+y\lambda_i < 1}}{Z_i(x,y)} = \frac{\gamma}{(1-x-y\lambda_i)^{m_1}} + \frac{\gamma''}{(1-x-y\lambda_i)^{m_2}} + O(1)$$

$$\text{(34b)} \qquad \underset{\substack{x+y\lambda_i \to -1 \\ x+y\lambda_i > -1}}{Z_i(x,y)} = \frac{\gamma'}{(1+x+y\lambda_i)^{1-m_1}} + \frac{\gamma'''}{(1+x+y\lambda_i)^{1-m_2}} + O(1) \; .$$

The choice of K_1 and K_2 which removed a singularity of a complex invariant eliminates the corresponding singularities in the real invariants. The value of any Riemann invariant on its characteristic is zero unless the characteristic ends on the body, and then its value is given by (17) or (33). Since each physical variable is a linear combination of all the Riemann invariants, each variable has the composite behavior implied by (32) and (34). Hence, as one approaches a characteristic leaving a tip the behavior anywhere in the fluid is given by (34). As one approaches a tip itself the behavior is the sum of terms from (32) and (34).

An expression of the form $1/|1-x-y\lambda|^m$ is square-integrable on line segments or over areas only if $m < 1/2$.

Hence, the singularities in (34a) and (34b) from a given exponent are not both square-integrable. The singularities in the complex invariant, see (32), are square-integrable over areas provided $0 < m < 1$. The generalized Kutta-Joukowsky condition of the next section is based on these observations.

In the case $b = 0$, the variables ζ and w separate out and the Riemann invariants Z_5 and Z_6 may be ignored or considered separately. Now it is possible to obtain a linear relation between $F(x)$, $G(x)$ and known functions. Hence there is only one integral equation and one exponent m and constant K. Otherwise there are no changes from the above description.

IV. A Generalized Kutta-Joukowsky Condition

The present section deals with the choice of the constants K_1, K_2 necessary for a unique solution. In ordinary fluid dynamics a similar non-uniqueness occurs in the subsonic solution: in this case a single integral equation is solved $m = \frac{1}{2}$, and the solution contains one arbitrary constant. The Kutta condition requires smooth flow at a sharp trailing edge and is equivalent to the addition of circulation to the flow sufficient to cause a confluence of the rear stagnation point on the body with the trailing edge. The singularity at the leading edge remains in the linearized solution in general unless the body is shaped to remove it.

A generalization of the Kutta-Joukowsky condition to determine a unique solution in the magneto-fluid dynamic case is proposed here. The singularities in this case have exponents m and $1-m$, $(0 < m < 1)$, at the trailing and leading edges, respectively, where m may have two values in the general two-dimensional problem. The proposed condition is to require the square integrability of these variables over any finite region in the flow, in particular in the neighborhood of the line singularities. This condition uniquely specifies the two constants K_1 and K_2. In an exact solution such square integrals must exist as they represent the energy in a region or the forces acting there. If the proposed condition were not valid, then the linearized solution would be a bad approximation to the exact solution near the line singularities both pointwise and in the square integral sense. Thus, non-linear or dissipative phenomena would affect the gross characteristics of the flow and the linearized solution would have almost no meaning. Hence, if the linearized solution is to be taken as a valid approximation (except in certain regions) then it must be used in conjunction with the proposed condition. The weaker point and line singularities remaining in the flow are not expected to be good pointwise approximations to the correct flow there, and a non-linear, dissipative analysis may have to be done to improve the solution in these regions. Nonetheless, one might hope that the over-all characteristics of the linearized solution are reasonably correct, as in ordinary fluid dynamics where integral properties such as lift on the airfoil are predicted fairly accurately by the linearized theory. However, experimental results will be the final arbiter.

For the restricted problem a curve in the hyperliptic region was found which separates regions $m > \frac{1}{2}$ from $m < \frac{1}{2}$. On the two sides of this critical curve, m has the values 1 and 0. It is instructive to follow the behavior of the singularities as the critical curve is crossed, say, from a region $m < \frac{1}{2}$ to a region $m > \frac{1}{2}$ (e.g., by varying $\overline{u}_0$). In the region $m < \frac{1}{2}$ the leading edge singularity is removed. As the critical curve is approached the trailing edge singularity tends to zero and at the critical curve there is no singularity at either edge. On entering the region $m > \frac{1}{2}$, m is close to unity, the trailing edge singularity is removed and the leading edge singularity is small. The total effect of crossing the critical curve is that the singularity disappears from one edge and reappears at the other. It is noted that this is a continuous process and is dependent on the Kutta condition used. If, say, a condition that the trailing edge singularity be removed were used instead, then there would be a discontinuous disappearance of the singularity at the leading edge across the critical curve.

A Kutta condition having similar effect to the one described above has been obtained by Stewartson(6) for the restricted problem in the incompressible limit. The removal of one of the edge singularities in either region $m > \frac{1}{2}$ or $m < \frac{1}{2}$ is related to its removal on the relevant side of the critical curve. There, the singularity at one of the edges has the limit unity for its exponent and when $m = 0$ (or 1) there are infinite forces on the body in the neighborhood of the leading (or trailing) edge. Since such a solution is unacceptable, Stewartson, by a continuity argument, removes that singularity whose exponent tends to one as the critical curve is approached from the relevant side. Thus Stewartson's condition produces the same result as the one proposed here, but it is imposed for different reasons. The argument here is based on the unacceptability of the solution without the proposed Kutta-Joukowsky condition without recourse to any continuity arguments. The distinction is clear in the general two-dimensional case where Stewartson's criterion is inapplicable as computations have shown that the two exponents m_1 and m_2 never become zero or one.

The singularities considered previously are those arising in the solution to the homogeneous integral equation. The shape of the body near the leading and trailing edges also determines the singular parts of the solution. For convenience, the edge of the body is now taken at the point $x = 0$ and the body oriented to lie in $X > 0$. In a neighborhood of $x = 0$ the body is assumed to be represented on top and bottom by the functions $y = h_1(x)$, $h_2(x)$ of the form $h_i(x) = x(c_{i1} + xR_{i1}(x)) + x^r(c_{i2} + xR_{i2}(x))$ where $0 < r < 1$ and R_{ij} are regular functions of x. If $c_{i2} \neq 0$ the term $c_{i2}x^r$ dominates at an edge and represents a blunt body.

The inhomogeneous term in the integral equation, depending linearly on the slope of the body, has the form

$$f(x) = d_1(1 + xR_1(x)) + x^{r-1}d_2(1 + xR_2(x))$$

where $R_i(x)$ is regular. The constants d_i are scalars or matrices according to whether a single equation or a system is considered. With cx^{-m} representing the singular part of the non-trivial solution of the homogeneous equation, the solution of the inhomogeneous equation near the given edge is of the form

$$e_1/x^m + e_2/x^{1-r} + R(x) \ ,$$

where R is regular and e_1 is determined by the generalized uniqueness condition. If $m < \frac{1}{2}$, e_1 is determined by application of the condition at the other edge of the body. If $m > \frac{1}{2}$, e_1 is put equal to zero. In either case, the solution contains a singularity of order $(1-r)$. For the above arguments in favor of the proposed uniqueness condition to be valid, it must be assumed that $r > \frac{1}{2}$ (the case $1-r = m > \frac{1}{2}$ is dismissed as too coincidental), and hence some blunt bodies are admissible. If $c_{i2} = 0$ $(i = 1, 2)$ so that the body is not blunt, the term in $1/x^{1-r}$ is absent.

V. Classification of the Singularities.

The exponent m of the singular terms involved in the solution of the integral equations (26) are complicated functions of the unperturbed velocity and magnetic fields $\overline{u}_o$, $\overline{B}_o$ and the Alfven and sound speeds A_o, a_o. Numerical calculations were performed to investigate the variation of m with these parameters.

In the restricted two-dimensional problem, m has a single value which may be obtained directly from the coefficient δ of the integral equation corresponding to (27) for F or G (recall that F and G are related). The formulae necessary for this calculation are set out in Appendix A. Computations were made to find m in the hyperliptic region (that is, over the whole range of values of u_o and θ) for a range of values of the ratio $\alpha = a_o^2/A_o^2$. When the coefficient δ is positive, $m = \frac{1}{\pi}\tan^{-1}\delta$, and $0 < m < \frac{1}{2}$. When δ is negative, $m = 1 + \frac{1}{\pi}\tan^{-1}\delta$, and $\frac{1}{2} < m < 1$. Hence, where δ changes sign, m jumps from zero to unity; this curve separates regions of $m < \frac{1}{2}$ from $m > \frac{1}{2}$ and is called the critical curve.

Figures 2 to 7 indicate the variation of the critical curve in the restricted problem over the range of values of α, and various contour lines of m are also shown. The figures represent one quadrant of the hyperliptic region and the complete diagram can be obtained by reflection in the axes. The unperturbed magnetic field, $\underline{B}_o$, is taken along the horizontal axis (as in Fig. 1). The velocity vector, $\underline{u}_o$, extends from the origin, and the size and inclination of this vector with respect to the other parameters then determine m and on which side of the critical curve it lies. Flows with $\bar{u}_o$ and $\bar{B}_o$ perpendicular always lie in the region $m > \frac{1}{2}$. For large α the critical curve is a continuous curve from the origin to the cusp on the axis of the inner wave front. The region $m < \frac{1}{2}$ increases in area as α decreases and splits into two branches as it overlaps with the outer limit of the hyperliptic region (Figure 4). The branch to the inner wave front is lost as the limit $a_o^2 = A_o^2$ is passed and for $\alpha < 1$ the critical curve extends from the origin to the point $(\sqrt{a_o^2 + A_o^2})$ on the perpendicular to $\bar{B}_o$. Figure 2, corresponding to the incompressible limit (A_o fixed, $a_o \rightarrow \infty$), shows Stewartson's result that the critical curve is given by

$$M_o^2 = \cos 2\theta \ . \tag{35}$$

Alternatively it represents the limit of vanishing magnetic field ($A_o \rightarrow 0$, a_o fixed) in which case the $m < \frac{1}{2}$ region is very small and the Kutta-Joukowsky condition of ordinary fluid dynamics is recovered as a limit of the generalized uniqueness condition.

The results just obtained may be checked with information inferred from certain expansions of δ. For small values of u_o the coefficient δ of the integral equation in the restricted case has the behavior

$$\delta \sim \alpha m_o \cos 2\theta \Big/ \left\{ \sin \theta \sqrt{1+\alpha} \right\} \ . \tag{36}$$

Thus the critical curve has slope $\frac{\pi}{4}$ at the origin. In the limit $\theta \rightarrow 0$, $\bar{u}_o$ arbitrary but fixed,

$$(37) \qquad \delta \sim 2\alpha m_o(1-m_o^2)\left(\frac{1}{1+\alpha} - m_o^2\right) \Big/ \left\{\theta C\sqrt{1+\alpha}\ [\alpha^2 m_o^2+(1+\alpha)(1-m_o^2)^2]\right\}$$

where

$$(38) \qquad C = \left\{(1-m_o^2)(1-M_o^2)/(1- {}^{2}_{o} - M_o^2)\right\}^{1/2} .$$

Hence δ changes sign twice, once at $\bar{u}_o = a_o$ and once at $\bar{u}_o = a_o A_o \sqrt{a_o^2 + A_o^2}$, the latter point being the position of the cusp of the inner wave front on the $\bar{B}_o$ axis. For $\alpha > 1$ (and hence $\theta = 0$, $\bar{u}_o = a_o$ lies on the outer wave front), δ has opposite signs in the hyperliptic region near $\theta = 0$ on the different sides of the cusp and δ vanishes at the cusp. As was observed in the figures, the critical curve leaves the origin and reenters the cusp. For $\alpha < 1$ (and hence $\theta = 0$, $\bar{u}_o = a_o$ lies on the inner wave front), in the hyperliptic region near $\theta = 0$, δ has only one sign. Now the critical curve leaves the origin and terminates at $\theta = \frac{\pi}{2}$, $\bar{u}_o = \sqrt{a_o^2 + A_o^2}$. Sears and Grad have shown that when $\bar{u}_o$ and $\bar{B}_o$ are parallel the system behaves as an ordinary fluid, while from (36) as $\theta \rightarrow 0$, $m \rightarrow + \frac{1}{2}$, the fluid dynamic result. Without much more care little more can be inferred from this treatment of the parallel flow case.

Numerical calculations to determine the exponents m_1, m_2 in the general two-dimensional case were also performed. The equations leading to the coupled integral equation (20) are given in Appendix B. The numerical results indicate

$0 \leq m_1 < \frac{1}{2} < m_2 \leq 1$, where $m_i \rightarrow \frac{1}{2}$ as $\theta \rightarrow 0$.

According to the proposed uniqueness condition, there is a singularity of exponent m_1 at the leading edge and a singularity of exponent $1-m_2$ at the trailing edge. Thus the leading edge singularity is stronger or weaker than the trailing edge one as $m_1 + m_2$ is greater or less than 1. In figures 9-15 the curves $m_1 + m_2 = 1$ are exhibited for various values of the parameters. The unit of speed is the Alfven speed A_o. For b large the two branches of the critical curve do not fit conveniently on one scale, hence part of the figure is shown in a reduced scale on the left. In such cases the second branch is supressed in the right-hand figure. The inner wave front either is drawn or if it is too small its position is indicated by a square. In each figure one region is marked for which $m_1 + m_2$ is greater or less than one. In all cases, as the curves are crossed, the inequality changes.

For small values of b one exponent should be close to the one computed when $b = 0$ and the other should be near zero or one. Hence the curve on which $m = 0$ (or 1) for $b = 0$ should approximate the curve $m_1 + m_2 = 1$ for $b \neq 0$ but small. Thus Figures 9 and 13 are similar to Figures 6 and 3 respectively. However in each case there is one additional curve when $b \neq 0$, but this curve vanishes as b goes to zero. Hence the distinctions present between $\alpha > 1$ and $\alpha < 1$ when $b = 0$ also occur for small values of $b \neq 0$. As shown in the figures 10, 11 and 12, as b increases the nature of the curves changes for $\alpha < 1$, so that the distinctions between $\alpha < 1$ and $\alpha > 1$

disappear.

Because of various difficulties in the numerical work certain features of figures 9-15 cannot be readily verified. First, it is not definitely known that the curves enter the origin, as shown . Secondly, it is not clear that the curves shown to intersect the outer surface of the ray cone do not continue very close to the cone and end at some other place. In both cases the numerical work strongly indicates the conclusions implied by the figures.

For large b , and m_o and M_o fixed, the coefficients μ, ν of the integral equation (26) are given by

$$\mu \sim \frac{M_o(\cos 2\theta - M_o^2)}{\sin\theta\ (1 + M_o^2)} + O(\tfrac{1}{b}) \tag{39a}$$

$$\nu \sim \frac{-m_o M_o^2 \sin\theta}{b\ (\cos 2\theta - M_o^2)} + O(\tfrac{1}{b^2}) \quad . \tag{39b}$$

An alternate description of the critical curve would be a curve across which $\mu + \nu$ changes sign. From (39a) and (39b) one branch of the curve for b large is seen to be

$$M_o^2 = \cos 2\theta \quad . \tag{40}$$

The other branch, found by setting m_o^2 and $M_o^2 = O(b^2)$, was computed to be

$$M_o^2 + m_o^2 \sin^2\theta = b^2 \tag{41a}$$

or, with $M_o^2 = \alpha m_o^2$,

(41b) $$M_o^2 = \frac{\alpha b^2}{\alpha + \sin^2\theta} .$$

For m_o^2 and $M_o^2 = O(b^2)$, the coefficients μ and ν of the integral equation (26) were found to be

(42a) $$\mu \sim \frac{m_o \sin\theta}{\sqrt{b^2 - M_o^2}} = \frac{m_o \sin\theta}{\sqrt{b^2 - \alpha m_o^2}}$$

(42b) $$\nu \sim \frac{\sqrt{b^2 - M_o^2}}{m_o \sin\theta} = \frac{\sqrt{b^2 - \alpha m_o^2}}{m_o \sin\theta}$$

where $M_o^2/b^2 = \alpha m_o^2/b^2 < 1$ is necessary if the velocity u_o is to remain in the hyperliptic region for large b. Without further terms in the expansion (41) it is not possible to decide whether this branch of the critical curve intersects the characteristic cone for large b.

VI. Flow near singular edges.

In order to understand the nature of the singularities present in this problem the flow quantities in the neighborhood of the leading and trailing edges have been computed numerically. The flows are also exhibited for singularities rejected by the proposed uniqueness condition. The flows suppressed by the uniqueness condition are marked accordingly. To facilitate computation only the restricted strictly two dimensional problem was considered. It suffices to consider the solutions to the homogeneous integral equations, i.e. functions representing flow past a flat body, since these solutions possess the same singularities as do solutions of the inhomogeneous problem. The leading and trailing edges may be treated separately, so that the body becomes the positive or the negative real axis with the origin at the leading or the trailing edge, respectively. Under these assumptions the flow becomes a similarity flow: the flow direction is the same at all points along a ray from the origin. A series of flows were computed numerically for a range of values of the parameters both in the $m > \frac{1}{2}$ and $m < \frac{1}{2}$ regions. Figures 16 to 23 show the shape of the streamlines of these flows near the leading and trailing edges.

The point singularity is taken in the form

$$(43) \qquad Z_2 = W_1 + iW_2 = (1 + i\gamma)\, Z^{-n}$$

where $n = 1 - m$ or m at a leading or trailing edge. The coefficient γ is determined in terms of the discontinuities across the body in W_1 and W_2. The complete details of the derivation of the flows computed in this section are given in Appendix C. The form (43) taken from Z_2 may be multiplied by a constant, thus affecting the flow variable. However, the ratio v/u, giving the streamline shape, is invariant. The only question outstanding is the positive or negative direction of the flow along the streamlines. The figures 16 to 23 must therefore be regarded as one of two possible flows, the second being the reversal of the one shown.

The flow quantities have infinite values along the characteristics from the leading and trailing edges, but the ratio v/u remains finite as these characteristics are approached. However, there is a discontinuous change in the streamline shape of the characteristic. In fact, it can be seen from the diagrams that these singular characteristics in a sense may act as line sources or sinks. That is, although the solutions do solve the linearized equations in a generalized sense, the linearized equations are a bad approximation to the non-linear ones in the neighborhood of the characteristics. Thus, the linearized equations represent there a bad approximation to mass conservation. It is possible that the introduction of shocks emanating from the leading and trailing edges may be needed. The regional effect of the line singularities of the linearized solution may provide a first approximation to the shock solution.

The flow diagrams may be interpreted as the hyperbolic flow carried along the characteristic superposed on the flow in the whole region near an edge due to the elliptic point singularity. The flow in the domain of dependence near an edge characteristic is dominated by the hyperbolic flow, but this influence weakens away from these characteristics. At points where the two constituent flows are opposing, asymptotes develop. The effect of the point singularity outside the domain of dependence is evident as a mixture of a circulatory flow around an edge and a flow developing asymptotes.

Appendix A.

The equations leading to the derivation of the integral equation in the hyperliptic restricted problem are set out. The real Riemann invariant is given by (16), which, with $b = 0$, may be written as $\underline{r} \cdot \underline{P}$ where

$$\text{(A1)} \qquad \underline{P} = \left(\frac{\sigma}{m_o^2}, v, \xi, \eta\right),$$

$$\text{(A2)} \qquad \underline{r}(\lambda) = \left\{(1-m_o^2)\cos\theta + \lambda\sin\theta,\ \ \lambda\cos\theta - \sin\theta,\right.$$
$$\left. 1 - \frac{m_o^2}{1+\lambda^2},\ \ \lambda\left(\frac{m_o^2}{1+\lambda^2} - 1\right)\right\}.$$

λ_1 is taken to correspond to the characteristic defining the domain of dependence in the lower half-plane and λ_4 in the upper. Let $\overline{c}$ be $\overline{r}(\lambda_1)$ and $\overline{d}$ be $\overline{r}(\lambda_4)$. The complex Riemann invariant Z_2 is obtained from (A1) with $\lambda = s + it$, and may be expressed in terms of its real and imaginary parts as

$$\text{(A3)} \qquad Z_2 = W_1 + W_2 = (\underline{a} + i\underline{b}) \cdot P$$

where

$$\text{(A4a)} \qquad \underline{a} = \left\{(1-m_o^2)\cos\theta + s\sin\theta,\ \ s\cos\theta - \sin\theta,\right.$$
$$\left. 1 - m_o^2(1+s^2-t^2)/D,\ \ s[m_o^2(1+s^2+t^2)-1]/D\right\}$$

$$\text{(A4b)} \qquad \underline{b} = \left\{t\sin\theta,\ \ t\cos\theta,\ \ 2st\ m_o^2 D,\right.$$
$$\left. t[m_o^2(1-s^2-t^2)-1]/D\right\}$$

(A4c) $$D = (1 + s^2 - t^2) + ds^2t^2 .$$

By the elimination procedures described one integral equation for the unknown function $F(x)$ may be obtained:

(A5) $$\mu\, F(x) + \frac{\nu}{\pi} \int \frac{F(\rho)}{\rho - x}\, d\rho = k(x)$$

where

(A6a) $$\mu = (a_1b_3-a_3b_1)(c_1d_4+c_4d_1) + (a_4b_1-a_1b_4)(c_1d_3+c_3d_1) + 2c_1d_1(a_3b_4-a_4b_3)$$

(A6b) $$\nu = (c_3d_1-c_1d_3)(a_1a_4+b_1b_4) - (c_4d_1-c_1d_4)(a_1a_3+b_1b_3) - (a_1^2 + b_1^2)(c_3d_4-c_4d_3)$$

and $k(x)$ is a known function.

Appendix B

Details of the completed integral equation in the general two-dimensional problem are given. The coefficients of the invariants associated with the quadratic factor are renamed to avoid confusion:

(B1a) $\bar{r}_+ = \bar{e}$

(B1b) $\bar{r}_- = \bar{f}$

The coefficients of the other two real invariants are also renamed:

(B2a) $\bar{r}_1 = \cos \phi_1 \bar{c}$

(B2b) $\bar{r}_4 = \cos \phi_4 \bar{d}$,

and for the complex invariants

(B3) $$\frac{\bar{r}_2}{\cos \phi_2} = (\bar{a} + i\bar{b}),$$

where $\bar{a}$ and $\bar{b}$ are real. It is easily verified that $\bar{a}$ and $\bar{b}$ are given by :

(B4a) $$\dot{a} = M_o^2\ s(1+s^2+t^2)/D - \sin\theta(\cos\theta + \sin\theta),$$

$$1-M_o^2(1+s^2-t^2)/D,\ [(1+s^2-t^2)(s\cos\theta - \sin\theta)$$

$$+ 2st^2\cos\theta]/D,\ - sa_3 + tb_3,\ bs(1+s^2+t^2)/D,\ b\sin\theta$$

(B4b) $$b = \Big\{ M_o^2\ t\ (1-s^2-t^2)/D - t\sin^2\theta,\ 2st/D,$$

$$[t\cos\theta(1-s^2-t^2) + 2st\sin\theta]/D,\ - (ta_3 + s\ b_3),$$

$$bt(1-s^2-t^2)/D,\ 0 \Big\}$$

The matrices A, B of (22) are given by

$$A_{11} = -2M_1c_6d_1 + M_2a_6b_1 - M_3a_6d_1 + 2M_4e_6f_1$$

$$A_{12} = 2M_1d_1(a_1c_6-a_6c_1)/b_1 - M_2a_1a_6 - 2M_4f_1(a_1e_6-a_6e_1)/b_1$$

$$A_{21} = -2N_1c_6d_1 + M_2a_6f_1 - N_3a_6d_1 + 2N_4e_6f_1$$

(B5) $$A_{22} = AN_1d_1(a_1c_6-a_6c_1)/b_1 - 2N_4f_1(a_1e_6 - a_6e_1)/b_1$$

$$B_{11} = - M_2a_1a_6$$

$$B_{12} = a_6(M_3d_1 - M_2b_1)$$

$$B_{21} = 0$$

$$B_{22} = a_6(N_3d_1 - M_2f_1)$$

The coefficients M_i in the above may be obtained from determinants of the matrix

$$(B6) \qquad M = \begin{pmatrix} c_3d_1 - c_1d_3 & c_4d_1 - c_1d_4 & a_6(d_1 - c_1) \\ a_3b_1 - a_1b_3 & a_4b_1 - a_1b_4 & a_6b_1 \\ a_3d_1 - a_1d_3 & a_4d_1 - a_1d_4 & a_6(d_1 - a_1) \\ e_3f_1 - e_1f_3 & e_4f_1 - e_1f_4 & e_6f_1 - e_1f_6 \end{pmatrix}$$

after the omission of the row indicated by the subscript. The matrix N is obtained from M by the replacement of the second row of M by

$$(a_3f_1 - a_1f_3 \ , \ a_4f_1 - a_1f_4 \ , \ a_6f_1 - a_1f_6) \ .$$

Appendix C.

The flows due to the point and line singularities near the leading and trailing edges are evaluated in the restricted problem. With the notation of Appendix A, the flow variables are obtained from the Riemann invariants from the equation

$$(C1) \qquad \underline{P} = R^{-1}\underline{Z} = \sum_{j=1}^{4} F_{ij} Z_j \quad ,$$

where

$$(C2) \qquad \underline{Z} = (Z_1, W_1, W_2, Z_4)$$

and where R is the 4×4 matrix whose rows are the vectors $\underline{c}$, $\underline{a}$, $\underline{b}$, $\underline{d}$.

With the homogeneous boundary conditions $v_t = v_b = 0$ and the usual conditions on the real invariants it follows that

$$(C3a) \qquad (Z_4)_t = - \quad f_{22}(W_1)_t + f_{23}(W_2)_t \quad /f_{24}$$

$$(C3b) \qquad (Z_1)_b = - \quad f_{22}(W_1)_b + f_{23}(W_2)_b \quad /f_{24}$$

The real invariants in the flow are given by their value traced back along a characteristic to the body. The flows for the leading and trailing edges are calculated separately and it is convenient to take each edge in turn as the origin of co-ordinates. Hence, the invariants have the associated values at any point in the flow given by

$$\left.\begin{aligned} &(\mathrm{C4a}) \qquad Z_1(x,y) = H(-y)\,H(-x-\lambda_1 y)\left\{Z_1(x+\lambda_1 y)\right\}_b \\ &(\mathrm{C4b}) \qquad Z_4(x,y) = H(y)\,H(-x-\lambda_4 y)\left\{Z_4(x+\lambda_4 y)\right\}_t \end{aligned}\right\}\begin{array}{l}\text{in the}\\ \text{trailing edge}\\ \text{calculation}\end{array}$$

$$\left.\begin{aligned} &(\mathrm{C5a}) \qquad Z_1(x,y) = H(-y)\,H(x+\lambda_4 y)\left\{Z_1(x+\lambda_4 y)\right\}_b \\ &(\mathrm{C5b}) \qquad Z_4(x,y) = H(y)\,H(x+\lambda_4 y)\left\{Z_4(x+\lambda_4 y)\right\}_t \end{aligned}\right\}\begin{array}{l}\text{in the}\\ \text{leading edge}\\ \text{calculation}\end{array}$$

where the body values of the invariants on the right-hand side are obtained from (C3) and H(x) denotes the Heaviside function.

The point singularity in the complex Riemann invariant according to (43) is

$$(\mathrm{C6a}) \qquad W_1 = r^{-n}(\cos n\psi + \lambda \sin n\psi)$$

$$(\mathrm{C6b}) \qquad W_2 = r^{-n}(\lambda \cos n\psi - \sin n\psi)$$

where r and ψ are defined by

$$(\mathrm{C7}) \qquad z = X + iY = x + sy + ity = re^{i\psi}.$$

The constant λ is determined from the discontinuities in W_1 and W_2 across the body. From the relation between F(x) and G(x) and the homogeneous boundary condition on v it follows that

(C8) $$b_1\left\{(W_1)_t - (W_1)_b\right\} = a_1\left\{(W_2)_t - (W_2)_b\right\} .$$

Hence, from (C7), for a trailing edge calculation $(\psi_t = \pi,\ \psi_b = -\pi)$

(C9) $$\gamma = -a_1/b_1$$

and for a leading edge calculation $(\psi_t = 0,\ \psi_b = 2\pi)$

(C10) $$\gamma = \frac{-a_1 \sin 2\pi n + b_1(1-\cos 2\pi n)}{a_1(1-\cos 2\pi n) + b_1 \sin 2\pi n} .$$

The flow variable vector $\underline{P}$ may now be obtained from the above equations. A point

$$x = k(\phi)\cos\phi \qquad y = k(\phi)\sin\phi$$

is chosen, and thus

$$\psi = \arctan\left\{t \sin\phi/(\cos\phi + s \sin\phi)\right\}$$

$$r = k\left\{(\cos\phi + s \sin\phi)^2 + t^2 \sin^2\phi\right\}^{1/2} .$$

The real and imaginary invariants are evaluated and substituted in (C1) for $\underline{P}$. The variables ζ and v are given directly and u is obtained from the relation (3). The ratio v/u for the streamline diagrams discussed in Section 7 is seen to be independent of k and r and is a function of δ only. A similarity flow is thus obtained.

Appendix D.

Specific solutions for flow quantities in the $\theta = \frac{\pi}{2}$ case are given in this appendix. From the symmetry properties for $\theta = \frac{\pi}{2}$ of the differential equation and boundary condition it is easily seen that if the body is antisymmetric (i.e., $(v)_t = (v)_b$), and if $(\sigma, v, w, \xi, \eta, \zeta)(x,y)$ is a solution of the boundary value problem, then $(-\sigma, v, w, \xi, -\eta, -\zeta)(x,-y)$ is also a solution. Hence v, w, and ξ are even functions of y and σ, η, and ζ are odd in y. Similarly, if $(v)_t = (-v)_b$, v, w, and ξ are odd in y and σ, η, ζ are even in y. It is evident that in general, the solution can be represented as the sum of two solutions, one depending only on the even part, $\frac{1}{2}[(v)_t + (v)_b]$, of the assigned values of v, and the other only on the odd part, $\frac{1}{2}[(v_t) - (v)_b]$. This decomposition corresponds to the diagonalization of the system (22).

An explicit representation for boundary values of the solution is given explicitly. It is supposed that the boundary values of v are decomposed as a sum

$$\text{(D1)} \qquad (v)_t = g + k$$

$$\text{(D2)} \qquad (v)_b = g - k ,$$

the solution is a sum of two functions. One, depending on g alone, represents the flow about a body with no thickness (cambered body) the other, depending on R alone, represents the flow about a symmetric body.

In the first case (cambered body) solution for boundary values of the density σ is given by

(D3) $$(\sigma)_t = -(\sigma)_b =$$

$$\frac{m_o^2}{\mu^2+\nu^2}\left\{\mu g(x) - \frac{\nu}{\pi}(1+x)^{m-1}(1-x)^{-m}\int_{-1}^{1}(1+s)^{1-m}(1-s)^{m}(s-x)^{-1}g(s)ds\right\}$$

$$+ c(1+x)^{m-1}(1-x)^{-m}$$

where

(D4) $$m = m_2 = 1 + \frac{1}{\pi}\tan^{-1}\frac{\nu}{\mu} \qquad \frac{1}{2} \leq m_2 < 1$$

(D5a) $$\mu = \lambda_1(1+\lambda_1^2-M_o^2)(1+b^2-b^2t^2)(\lambda_1^2+t^2)^{-1}(1+b^2+b^2M_o^2)^{-1}$$

$$+ M_o b^2(1+b^2+b^2M_o^2)^{-1}$$

(D5b) $$\nu = -t(1+b^2+b^2\lambda_1^2)(t^2+M_o^2-1)(\lambda_1^2+t^2)^{-1}(1+b^2+b^2M_o^2)^{-1}$$

and λ_1 , $-\lambda_1$, it, $-it$ are the characteristic roots of the linearized flow equations.

Since $m_2 > 1/2$, the uniqueness condition is applied at the trailing edge. The pressure difference across the body is given by

$$\text{(D6)} \qquad (p)_t - (p)_b = 2\rho_o u_o^2 \frac{(\sigma)_t}{m_o^2} \quad .$$

The solution (D3) in the case $b = 0$ with a Kutta condition applied at the trailing edge corresponds to the solution obtained by McCune using a Prandtl-Glauert transformation in conjuction with the incompressible theory of Sears.

The singularity in the solution (D3) tends to the ordinary fluid dynamics singularity $(m = 1/2)$ as the magnetic field tends to zero.

As an example, it is instructive to look at the solution (D3) for the flow past a flat plate at the incidence α , applying the uniqueness condition at the trailing edge. The solution gives

$$\text{(D7)} \qquad (p_t - p_b) = 2\rho_o u_o^2 \alpha(-\nu)(\mu^2+\nu^2)^{-1}(\sin \pi m_2)^{-1}\left(\frac{1-x}{1-x}\right)^{1-m_2}$$

The pressure singularity near the leading edge is reduced by the magnetic field since $m_2 > \frac{1}{2}$.

For the symmetric body, the solution for boundary values of the fluid component w of velocity is given by

D(8) $(w)_t = - (w)_b$

$$= - \frac{1}{\nu_1^2+\mu_1^2} \Big\{ \mu_1 K(x) + \frac{\nu_1}{\pi} (1+x)^{m-1}(1+x)^{-m} \int_{-1}^{1} (1+s)^{1-m}(1-s)^{m}(s-x)^{-1} K(s)\,ds,$$

$$+ c_1(1+x)^{m-1}(1-x)^{-m}$$

where

(D9a) $$\mu_1 = \lambda_1 M_o^{-1}\, b^{-1} + b(M_o^2+t^2)(1+\lambda_1^2)(t^2+\lambda_1^2)^{-1} \quad ,$$

(D9b) $$\nu_1 = b\lambda_1 t^{-1}(\lambda_1^2-M_o^2)(1-t^2)(t^2+\lambda_1^2)^{-1} \quad ,$$

(D9c) $$m = m_1 = \frac{1}{\pi} \tan^{-1} \frac{\nu_1}{\mu_1} \qquad \text{and} \quad 0 < m_1 < \frac{1}{2} \quad ,$$

(D9d) $$K(x) = \lambda_1 b^{-1}\, M_o^{-2}(1+b^2+b^2M_o^2)\zeta_o$$

$$+ \Big\{ (M_o^2+t^2)(1+\lambda_1^2-M_o^2)(t^2+\lambda_1^2)^{-1} - \lambda_1 M_o^{-1} \Big\}\, k(x)$$

$$+ \frac{\lambda_1}{\pi t} (\lambda_1^2-M_o^2)(1-M_o^2-t^2)(t^2+\lambda_1^2)^{-1} \int_{-1}^{1} k(\xi)(\xi-x)^{-1}\, d\xi,$$

and ζ_o is the constant value of ζ on the body. Here the Kutta condition is applied at the leading edge.

Remark

The two previous examples of the cambered and thickened bodies illustrate the general case in which $0 < m_1 \leq m_2 < 1$, and one singularity is left at each edge of the general body on application of the uniqueness condition. The point on the perpendicular axis to $\underline{B_o}$ at which the singularities become equal is given by

$$\mu_1\nu + \mu\nu_1 = 0 \quad . \tag{D10}$$

Footnotes

1. W.R. Sears and E.L. Resler, "Theory of Thin Airfoils in Fluids of High Electrical Conductivity", J. Fluid. Mech., 5, 257 (1959).

   J.E. McCune and E.L. Resler, "Compressibility Effects in Magnetoaerodynamic Flows Past Thin Bodies", J. Aero Space Sci., 27, 493 (1960).

2. See H. Grad.,"Reducible Problems in Magneto-Fluid Dynamic Steady Flows", Rev. Mod. Phys., 32, 830 (1960).

3. See W.R. Sears and E.L. Resler, op. cit.

4. A preliminary version of the uniqueness condition was given by the authors in J. Aero Space Sciences, 29, 244 (1962).

5. S.G. Mikhlin, Integral Equations (Pergamon Press, New York, 1957) 126.

6. K. Stewartson, "Magneto-Fluid Dynamics of Thin Bodies in Oblique Fields", Zeit für ang. Math. and Phys., 12, 261 (1961).

$$A_1^2 = A_0^2(1+b^2)$$

$$\text{with}\begin{cases} s = a_0^2 + A_1^2 \\ t = \sqrt{S - 4a_0^2 A_0^2}\ , \end{cases}$$

$$\tilde{A}_0^2 = \frac{s+t}{2}$$

$$\tilde{a}_0^2 = \frac{s-t}{2}$$

$$OD = OD' = \sqrt{\frac{a_0^2\ A_0^2}{a_0^2 + A_1^2}}$$

$$OF = OF' = \tilde{A}_0$$

$$OE = OE' = A_0$$

$$OG = OG' = \tilde{a}_0$$

$$OH = \sqrt{a_0^2 + A_1^2}$$

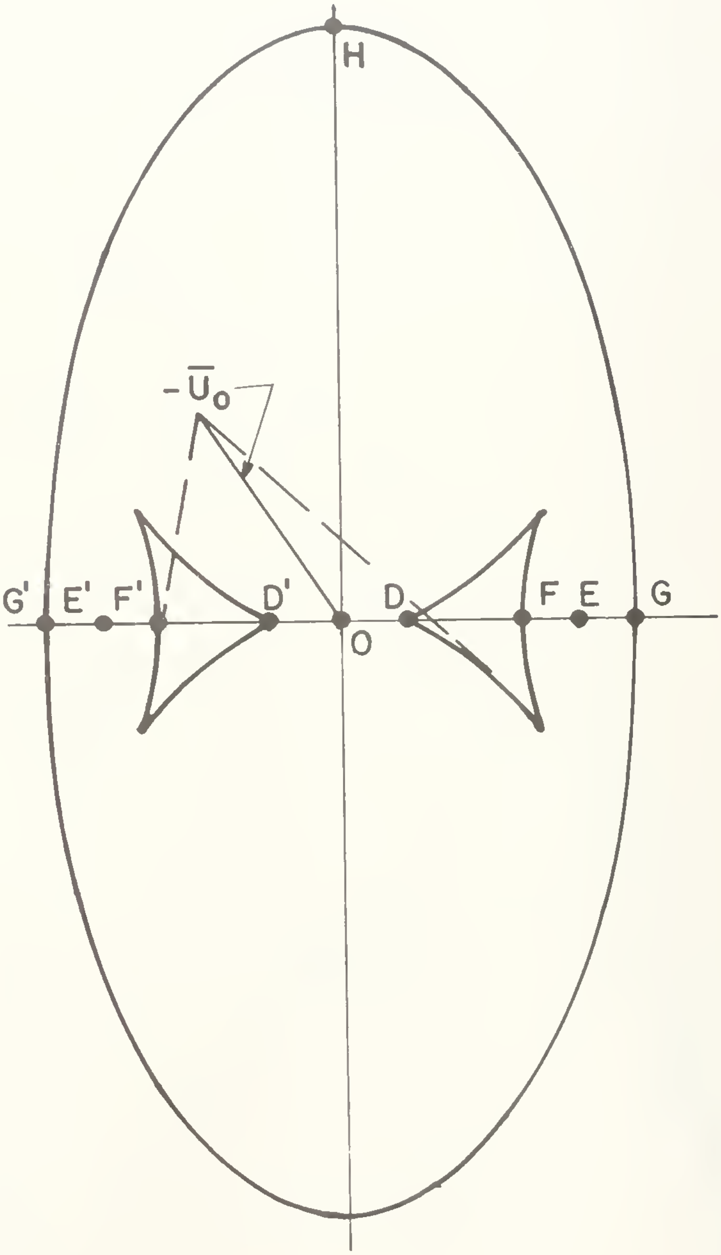

Figure 1: SECTION OF THE RAY CONE FOR A TWO DIMENSIONAL MAGNETOHYDRODYNAMIC WAVE.

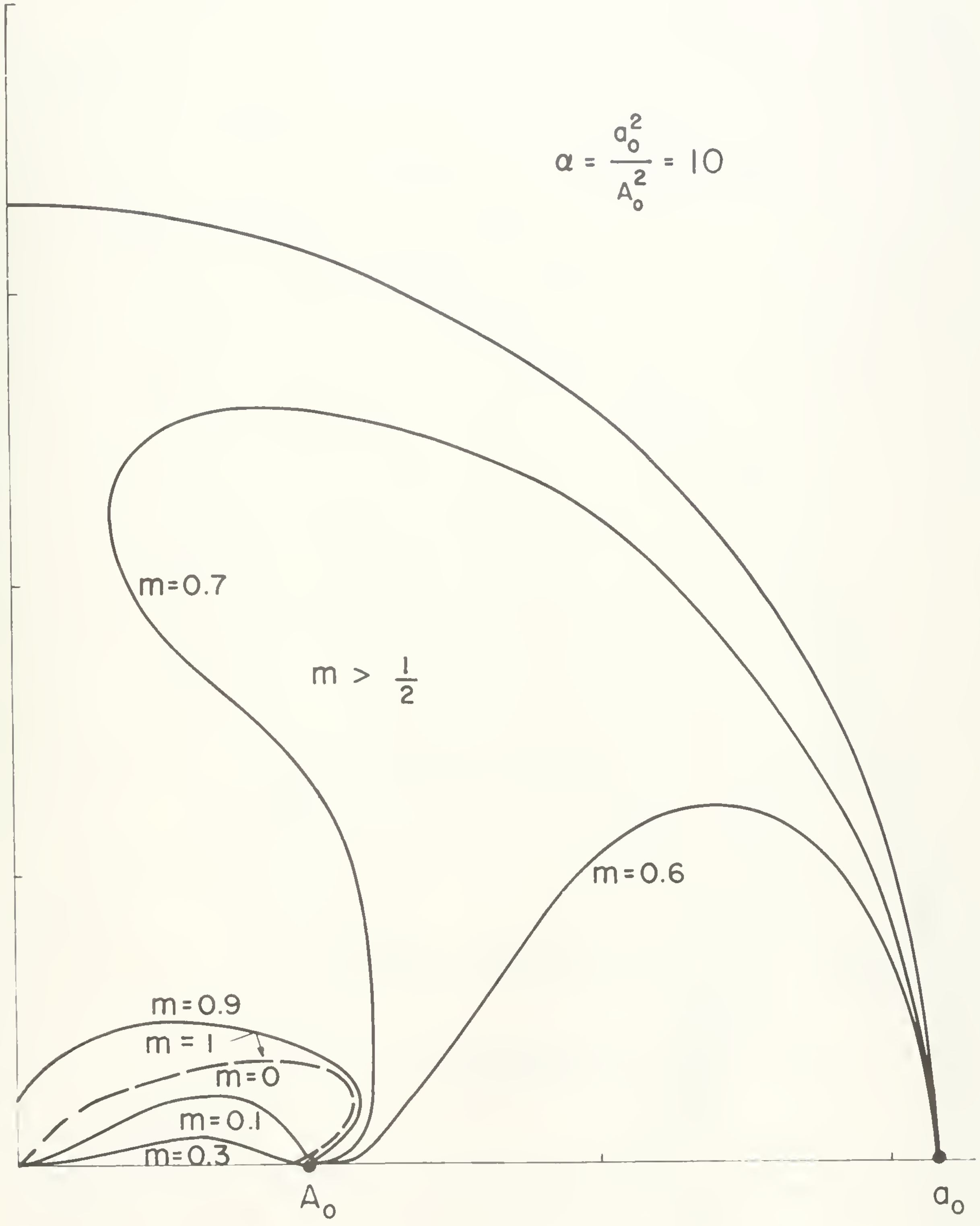

Figure 2: EXPONENTS m OF SINGULARITIES FOR $\alpha = 10$.

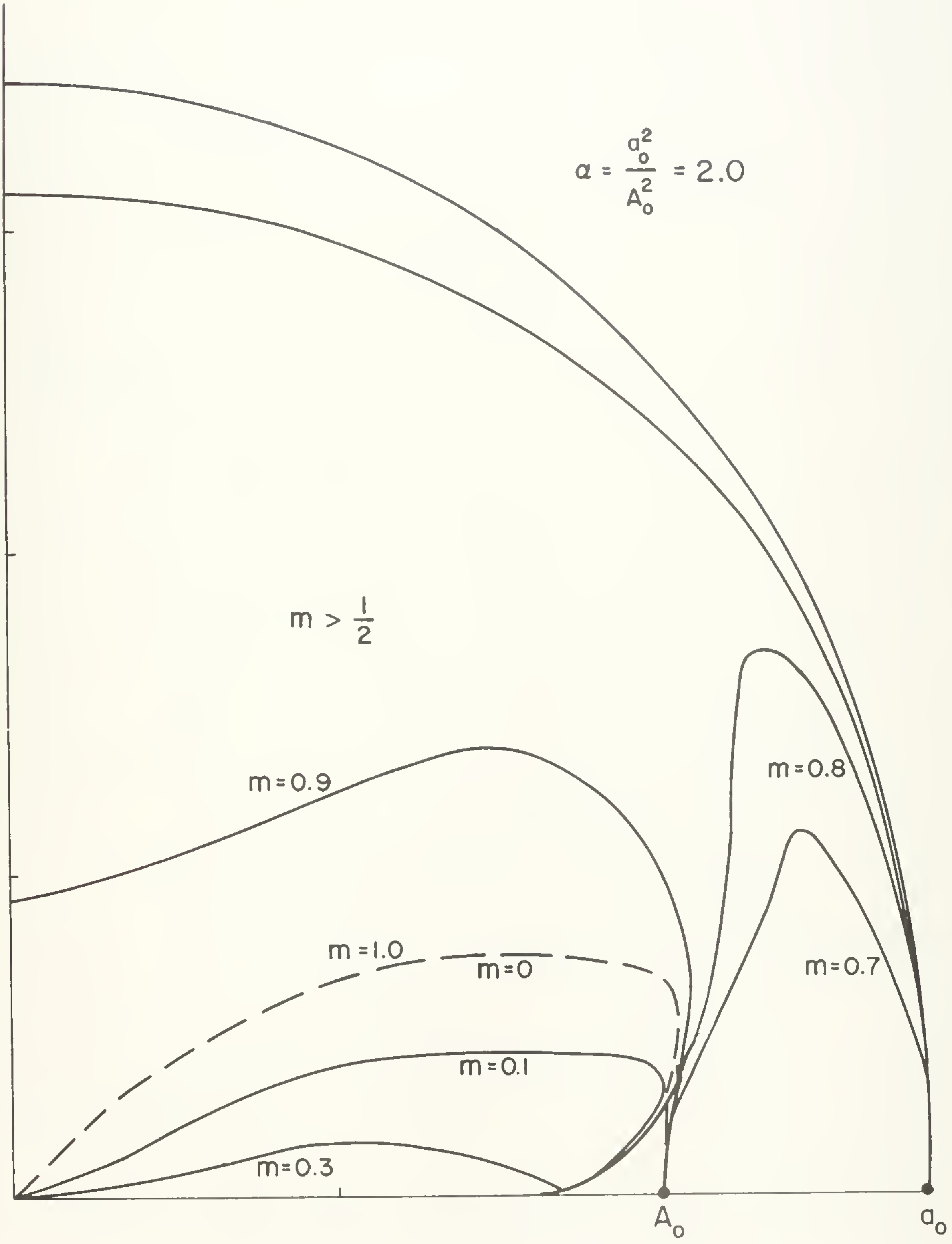

Figure 3: EXPONENTS m OF SINGULARITIES FOR $\alpha = 2$.

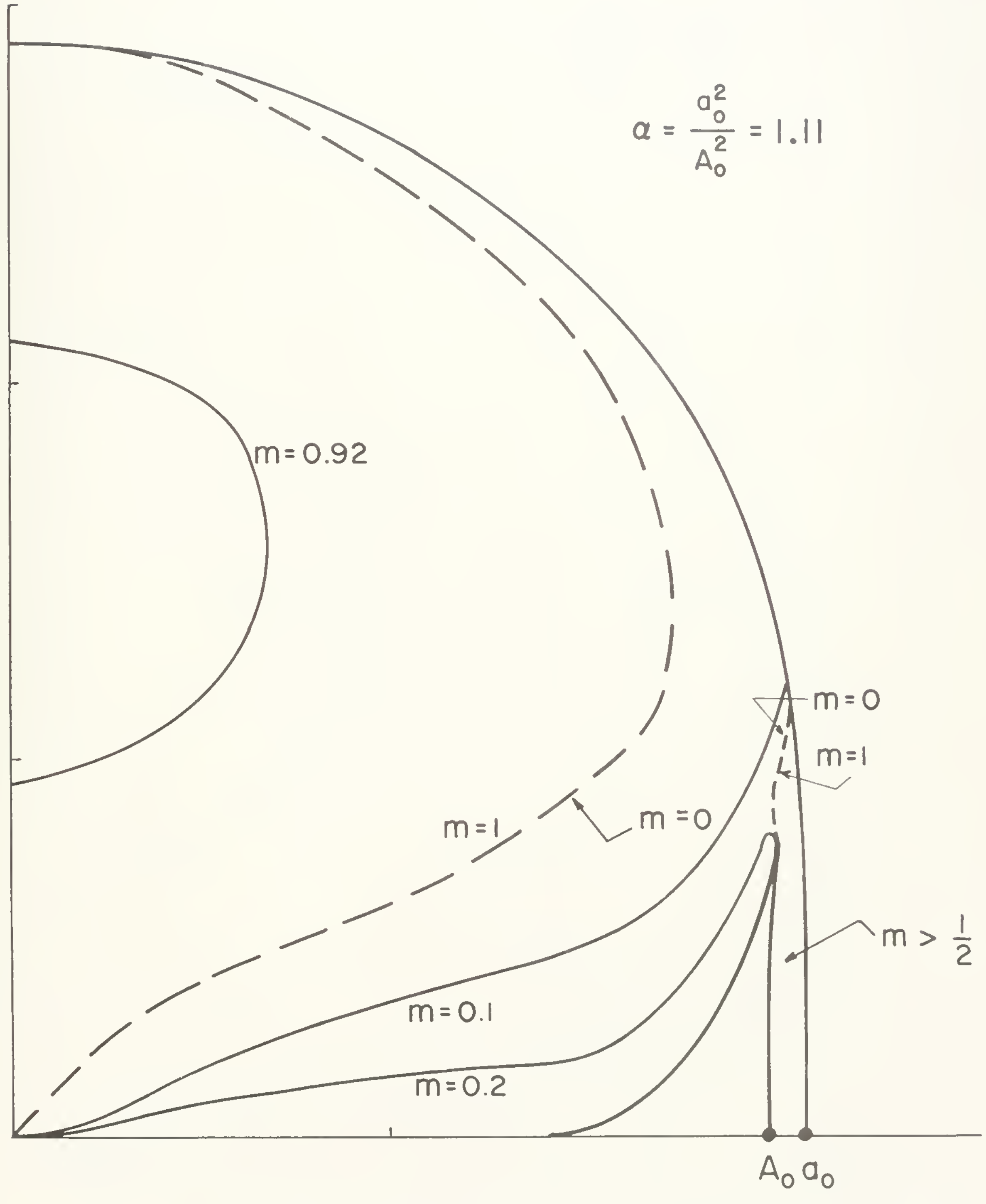

Figure 4: EXPONENTS m OF SINGULARITIES FOR $\alpha = 1.11$

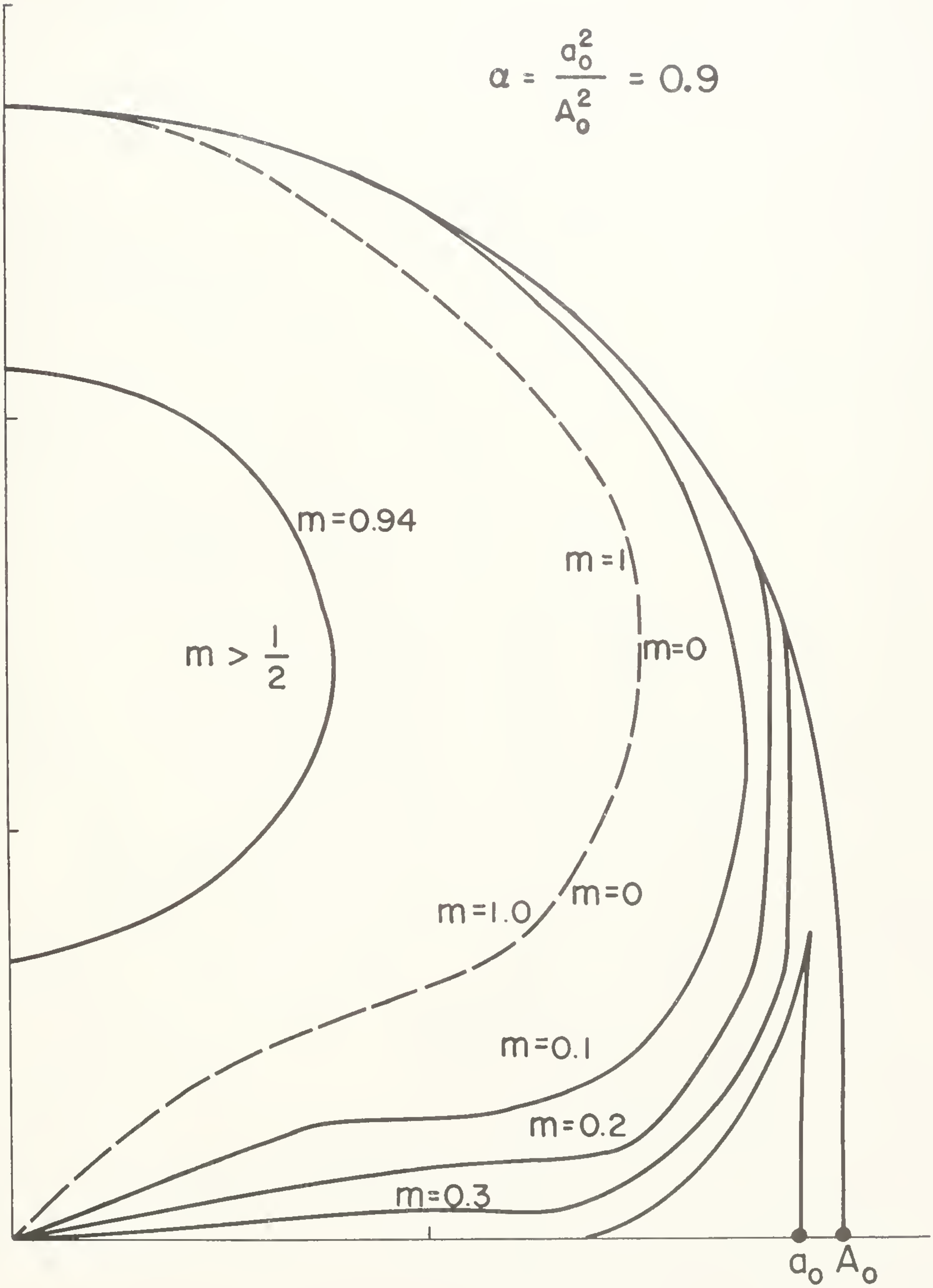

Figure 5: EXPONENTS m OF SINGULARITIES FOR α = 0.9.

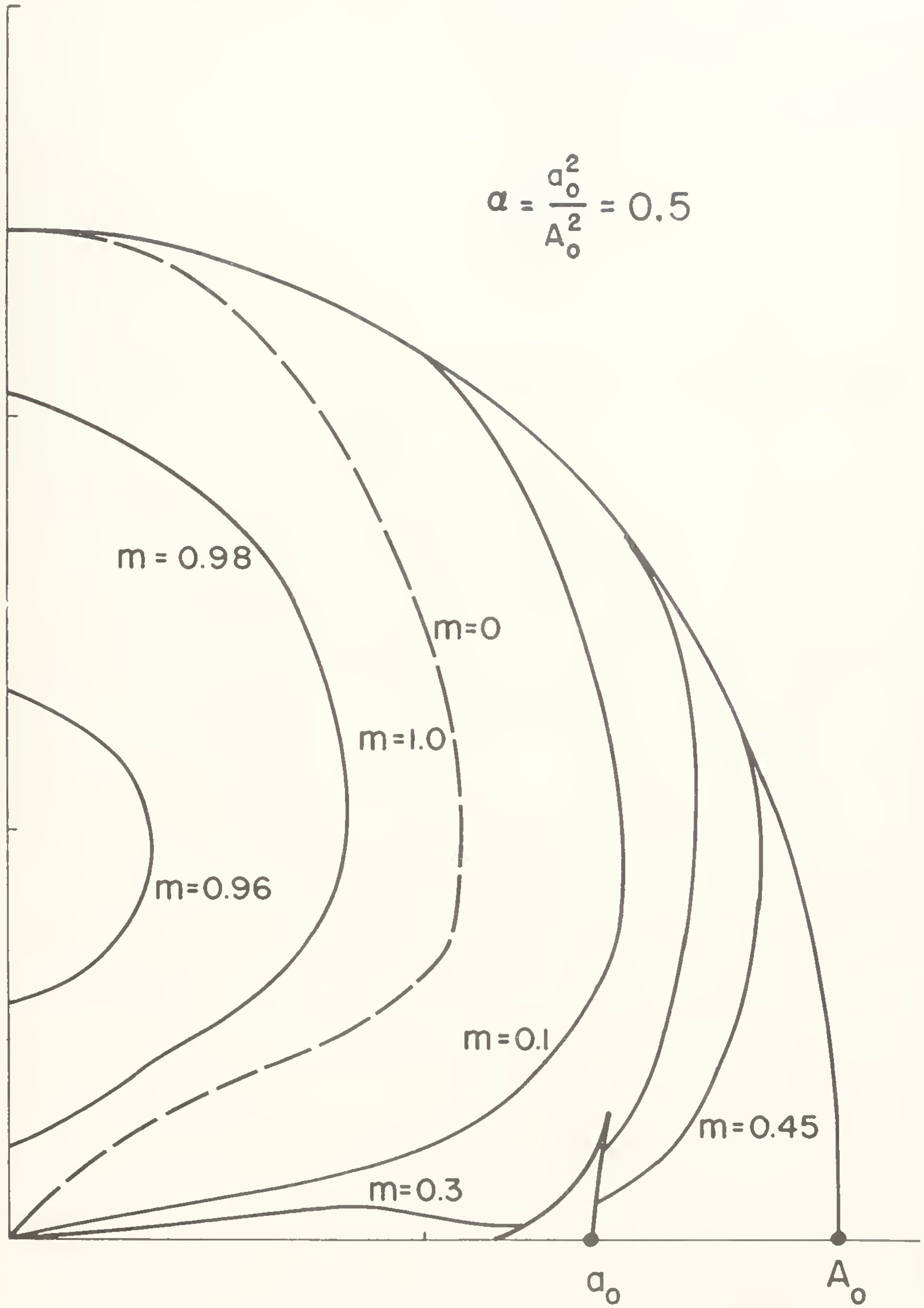

Figure 6: EXPONENTS m OF SINGULARITIES FOR $\alpha = 0.5$.

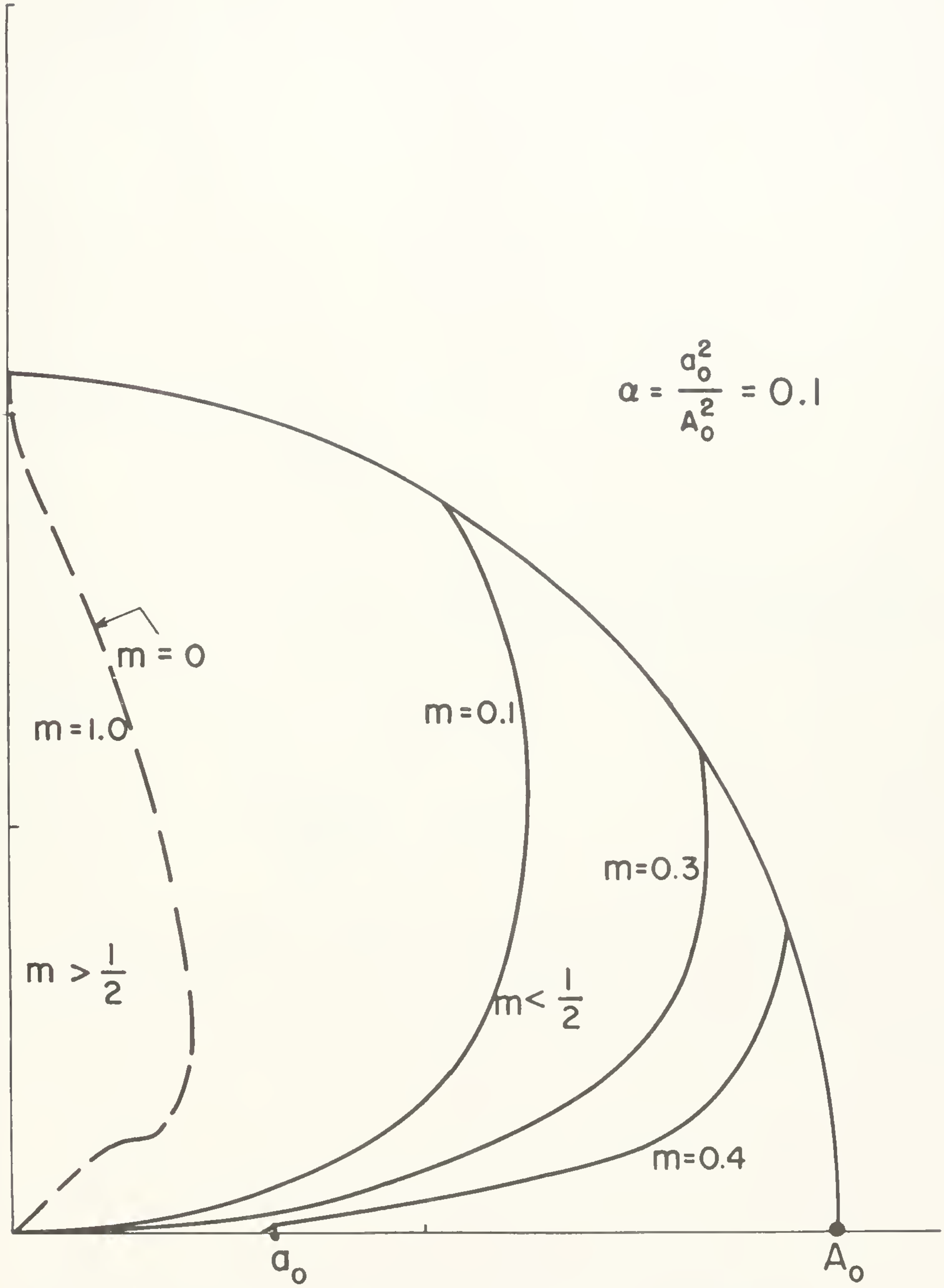

Figure 7: EXPONENTS m OF SINGULARITIES FOR $\alpha = 0.1$.

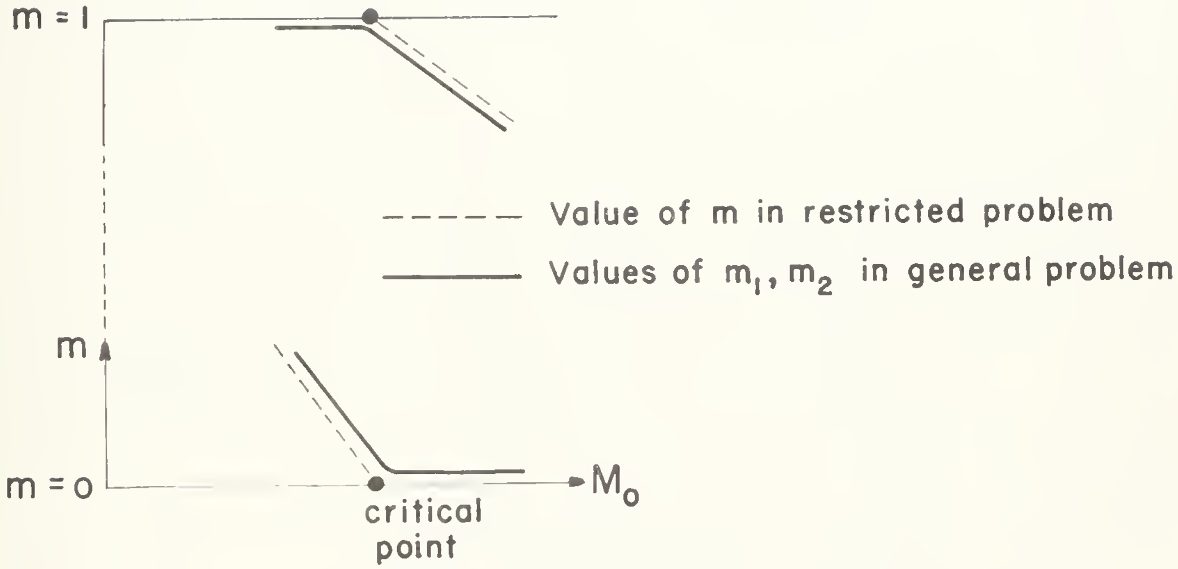

Figure 8: BEHAVIOR OF EXPONENENTS FOR SMALL b.

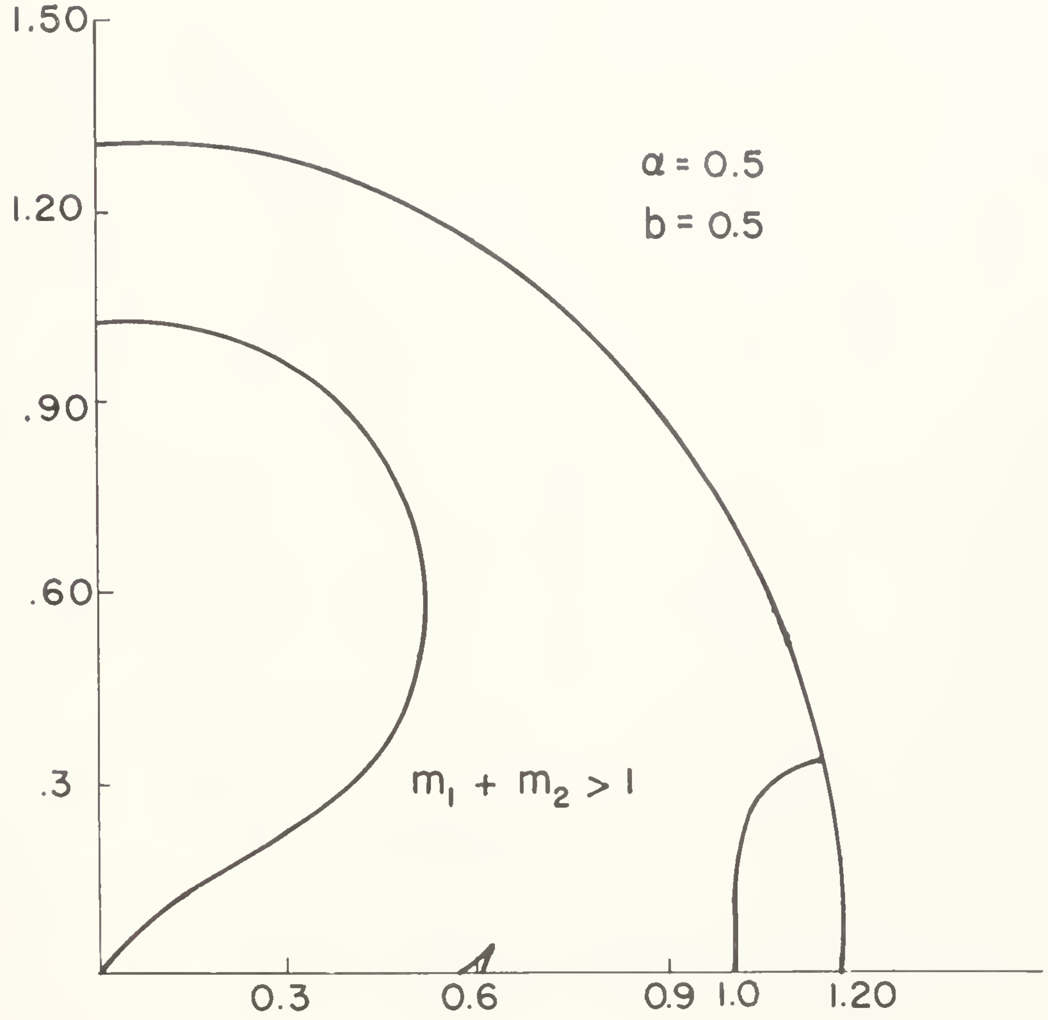

Figure 9: CRITICAL FOR $\alpha = 0.5$ $b = 0.5$

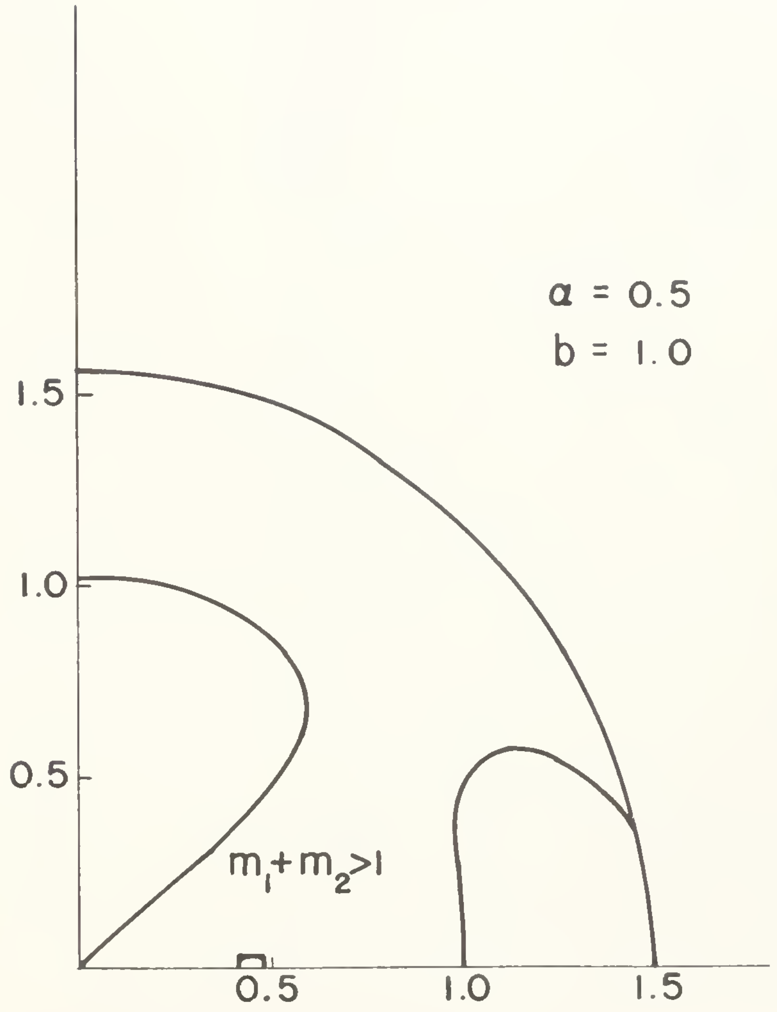

Figure 10: CRITICAL CURVES FOR α = 0.5 b = 1 0

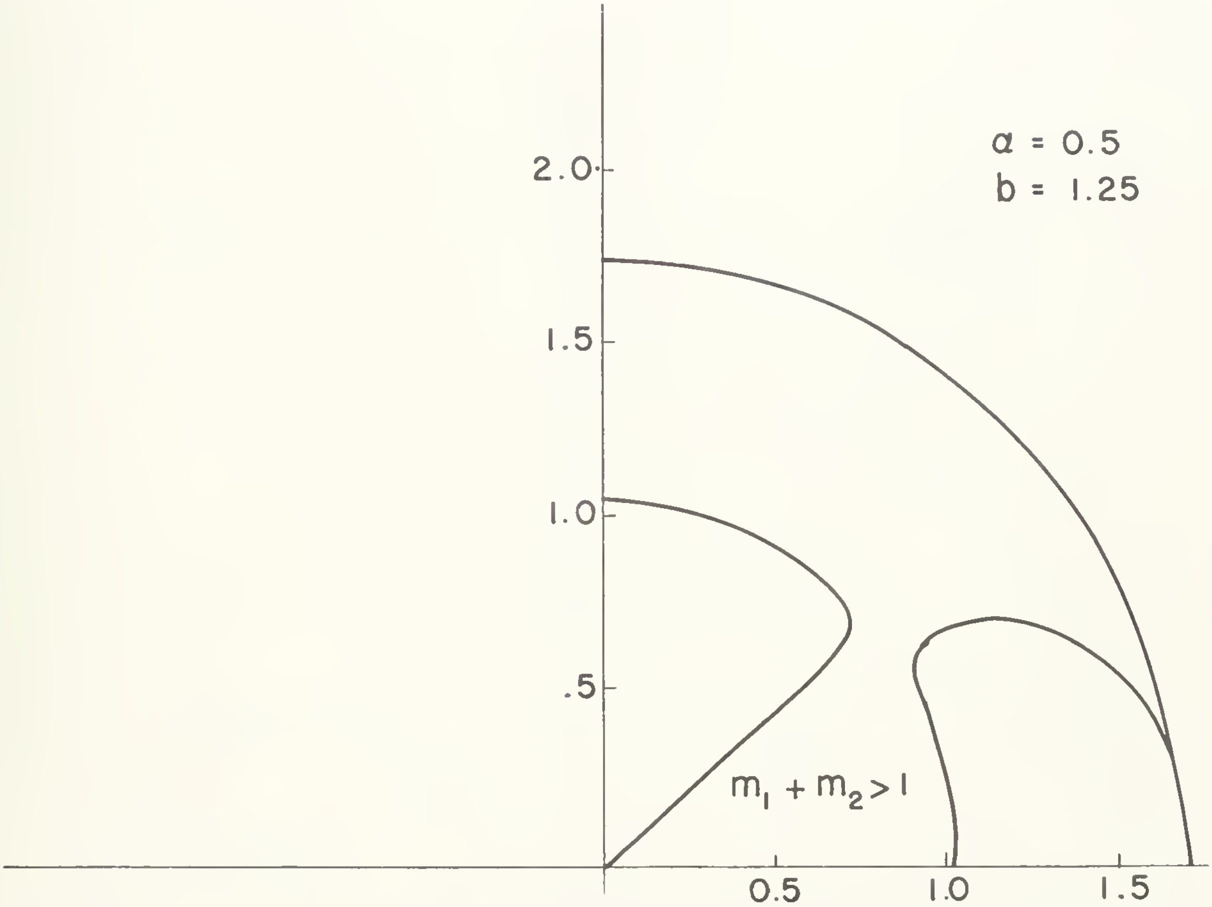

Figure 11: CRITICAL CURVES FOR $\alpha = 0.5$ $b = 1.25$

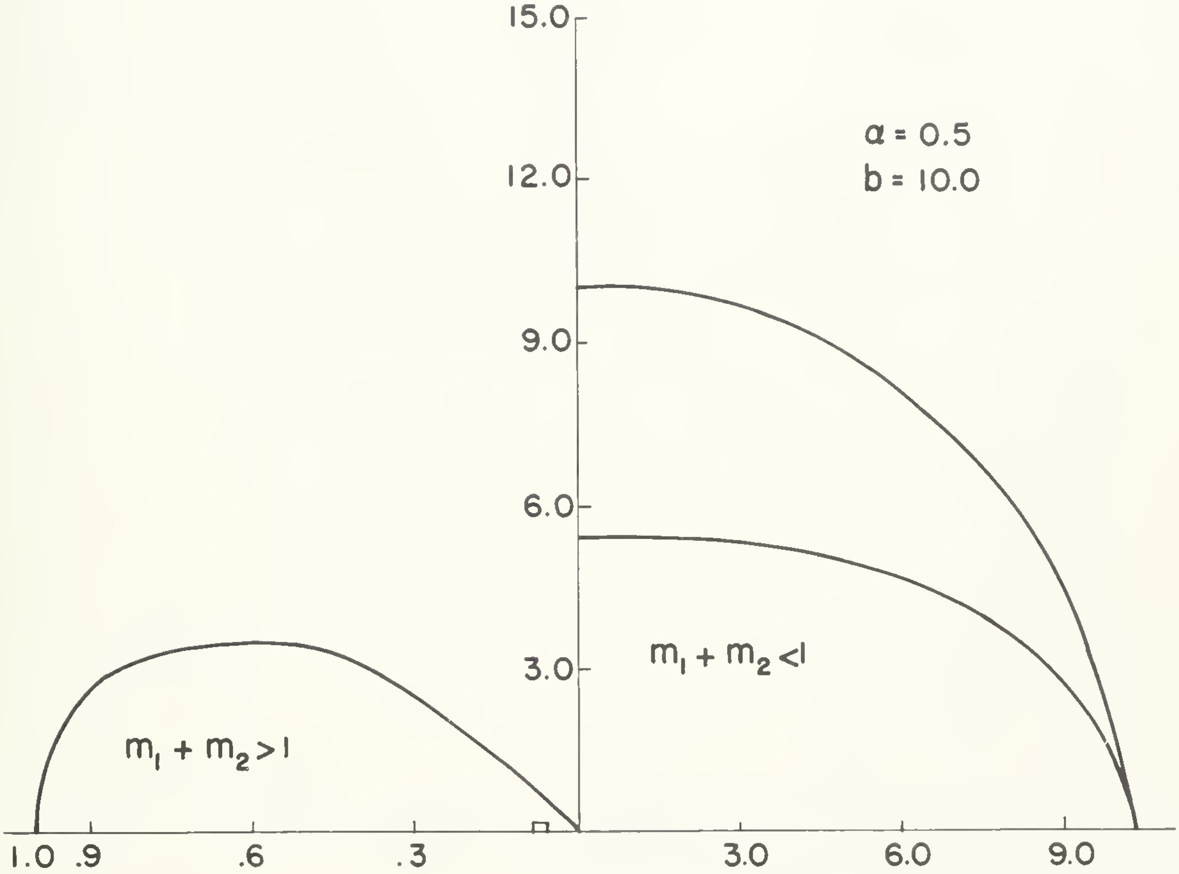

Figure 12: CRITICAL CURVES FOR α = 0.5 b = 10.0

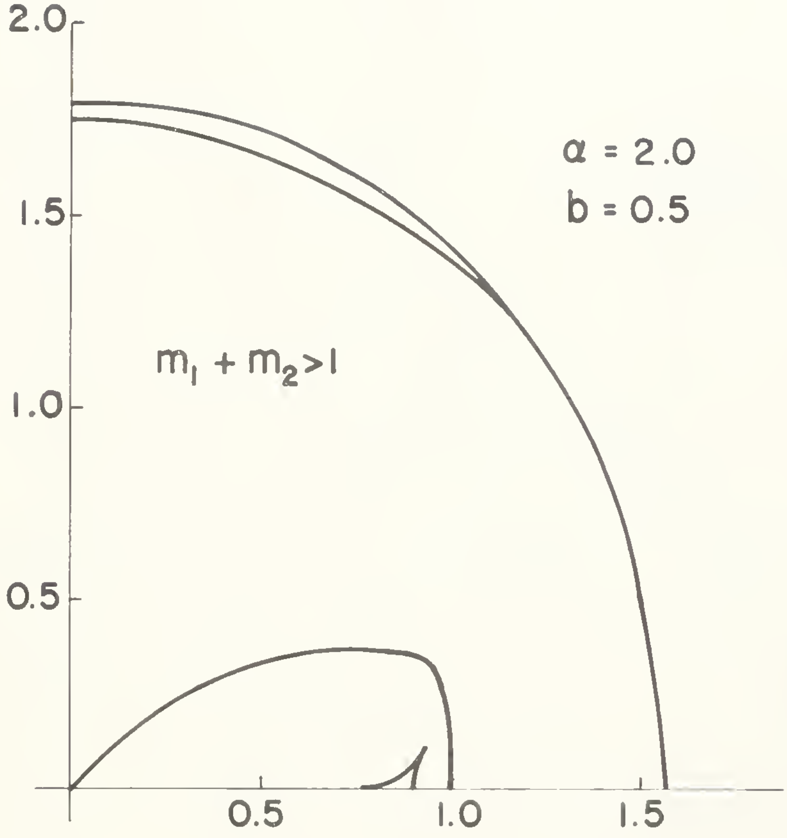

Figure 13: CRITICAL CURVES FOR $\alpha = 2.0$ $b = 0.5$

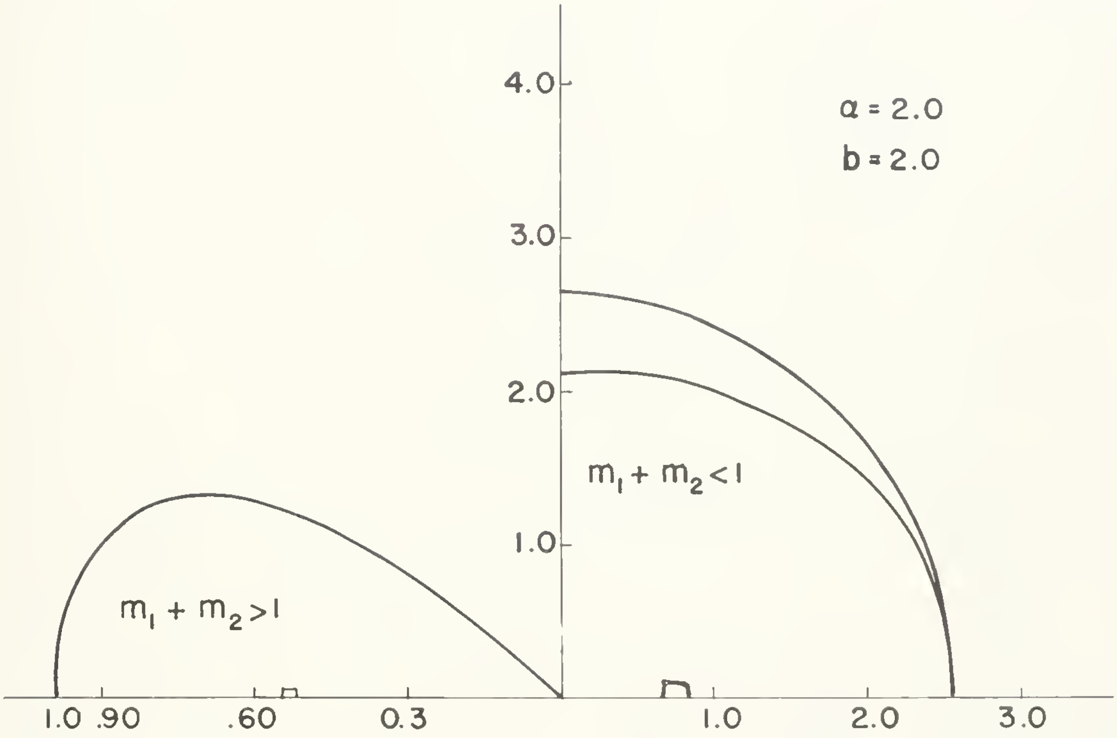

Figure 14: CRITICAL CURVES FOR $a = 2.0$ $b = 2.0$

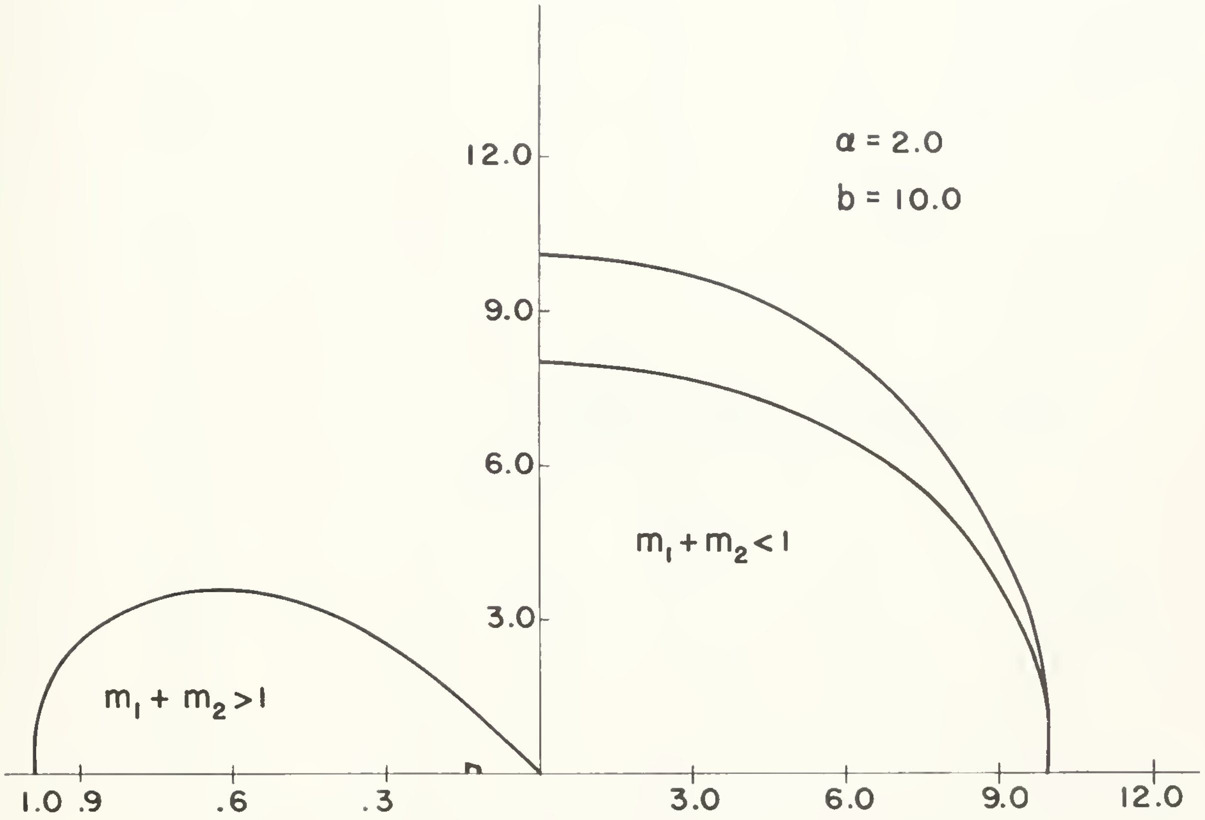

Figure 15: CRITICAL CURVES FOR $\alpha = 2.0$ $b = 10.0$

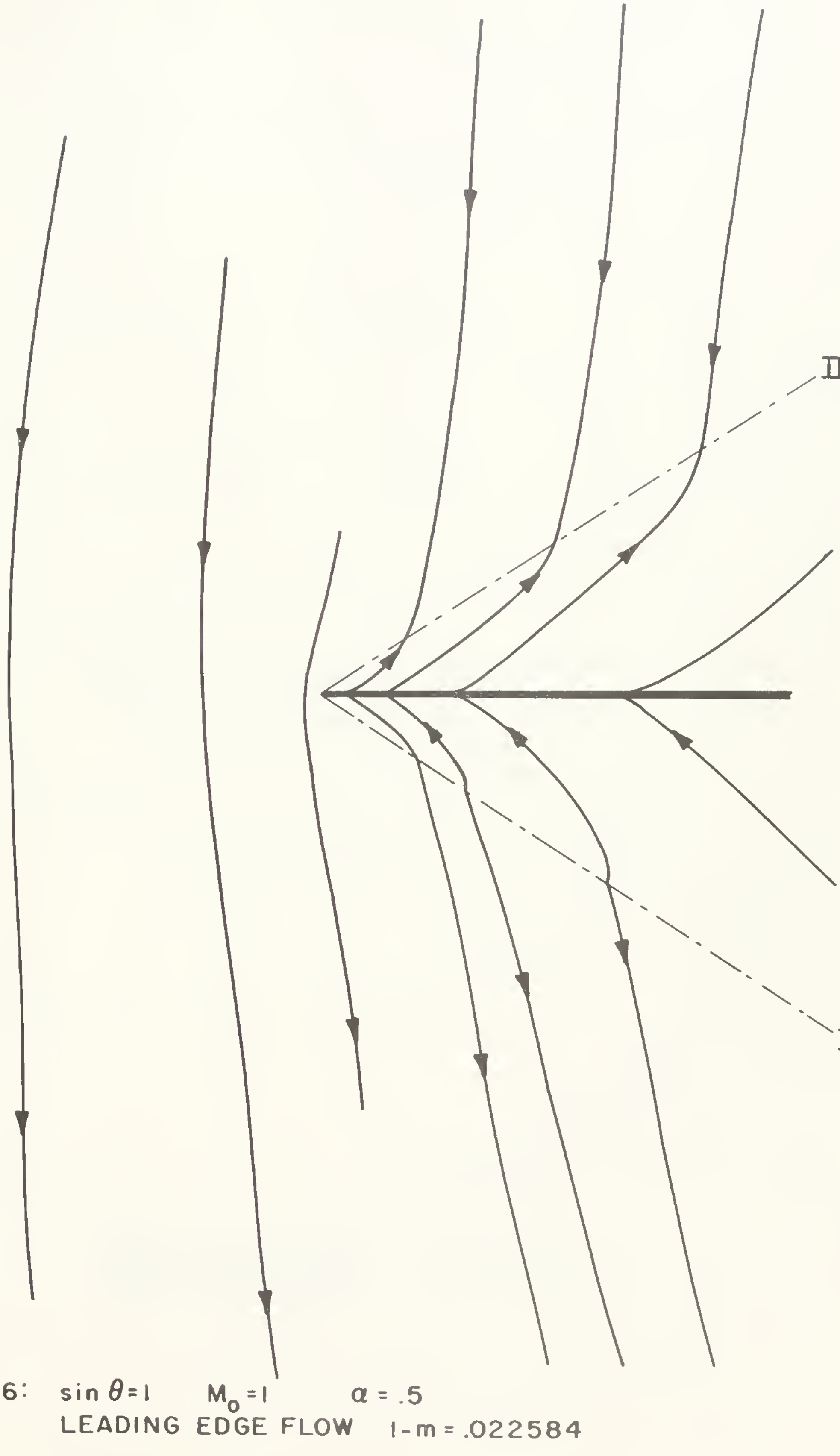

Figure 16: $\sin\theta = 1$ $M_0 = 1$ $\alpha = .5$
LEADING EDGE FLOW $1 - m = .022584$

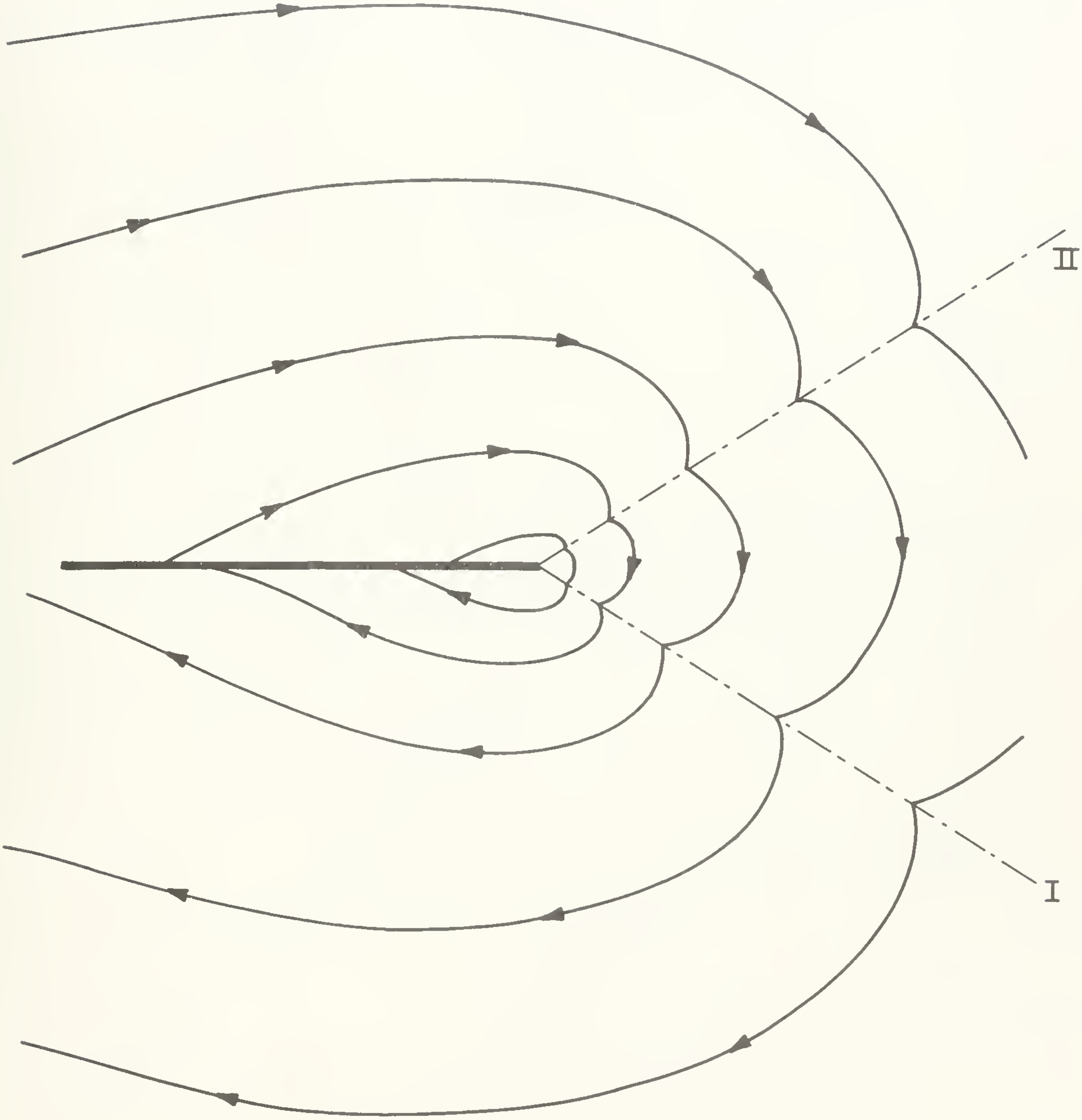

Figure 17: $\sin\theta = 1$ $M = 1.$ $\alpha = .5$
TRAILING EDGE FLOW $m = .977416$
FLOW SUPPRESSED

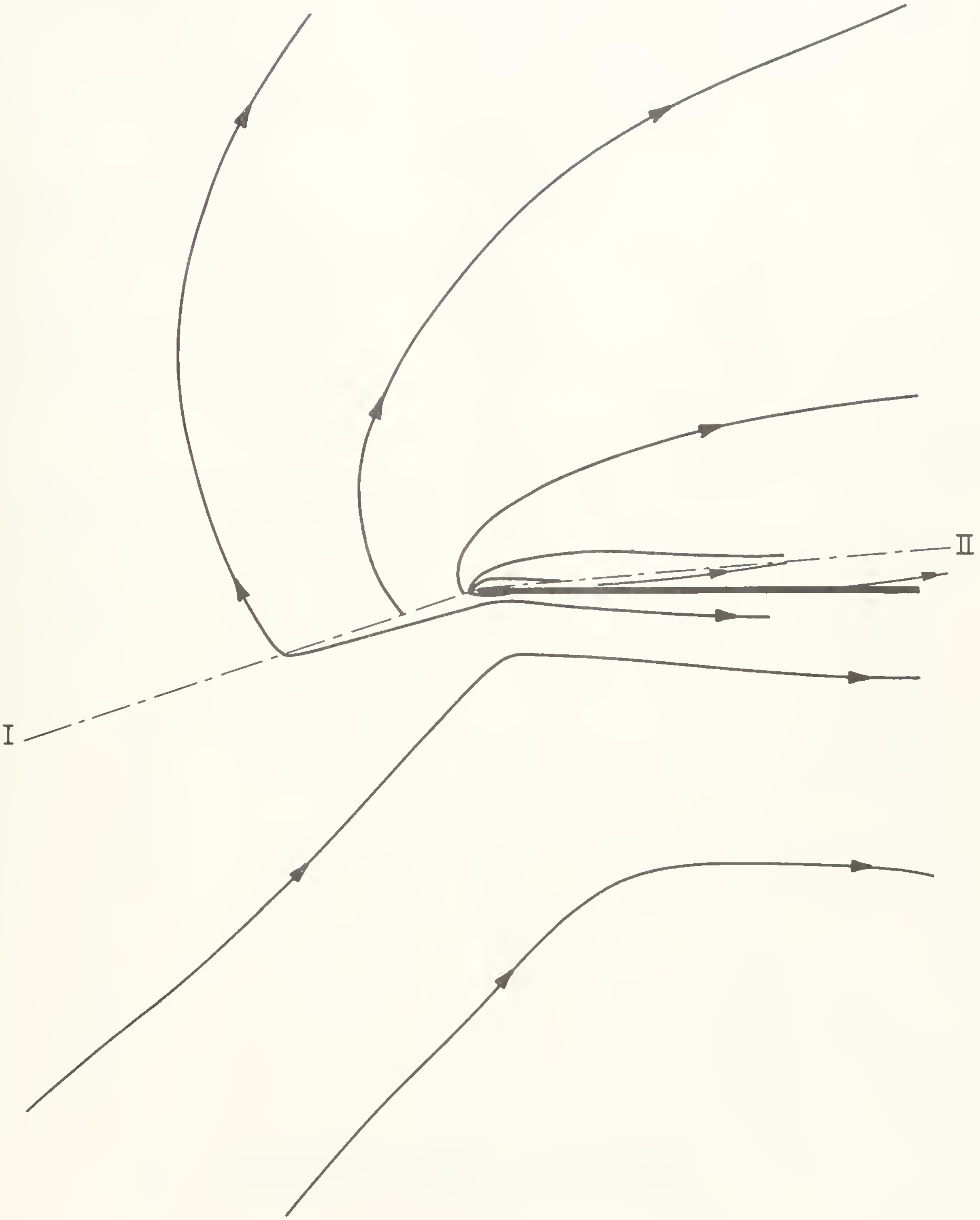

Figure 18: $\sin\theta = .15$ $M_0 = .45$ $\alpha = 2$
LEADING EDGE FLOW $1 - m = .68842$
FLOW SUPPRESSED

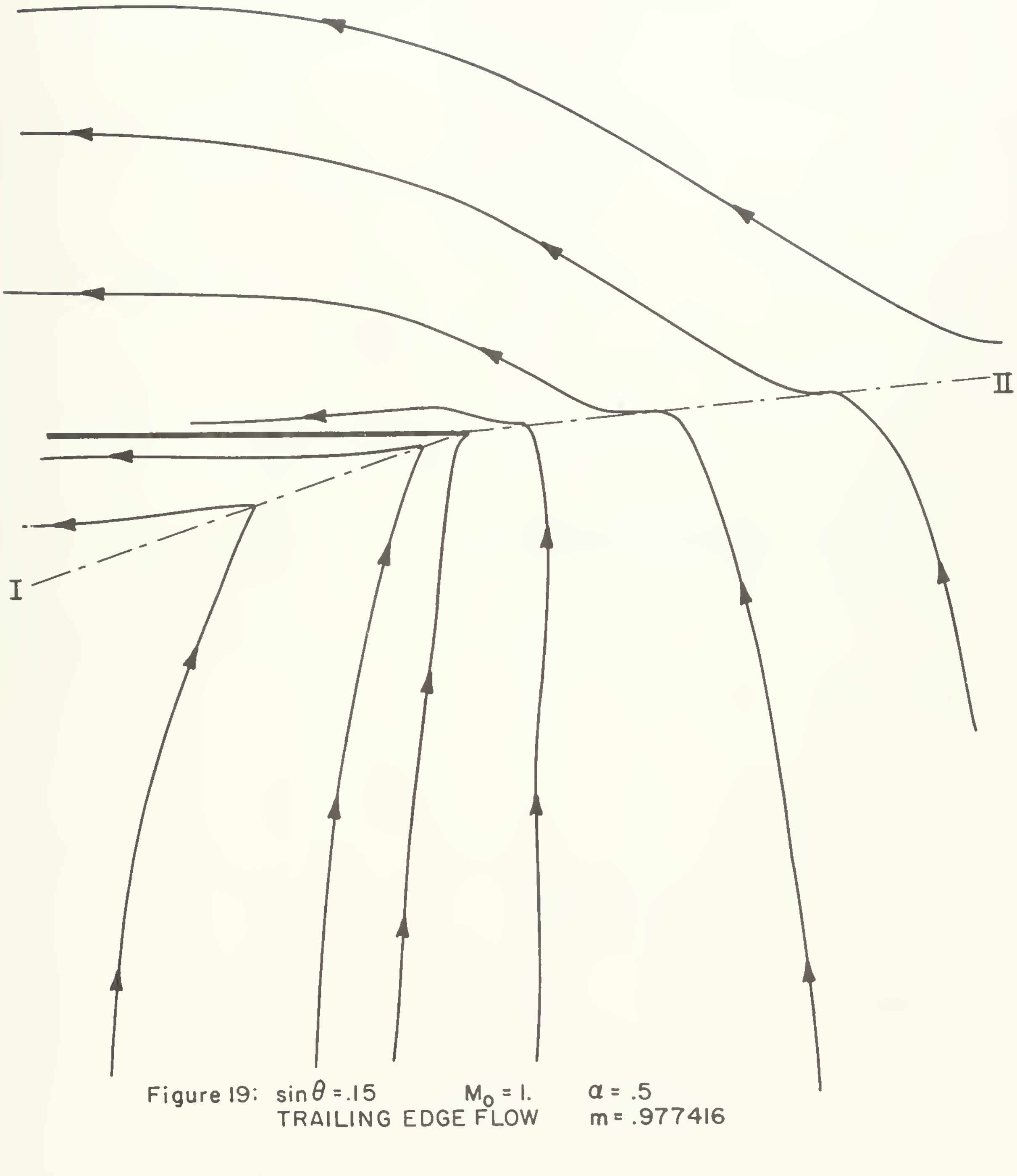

Figure 19: $\sin\theta = .15$ $M_0 = 1.$ $\alpha = .5$
TRAILING EDGE FLOW $m = .977416$

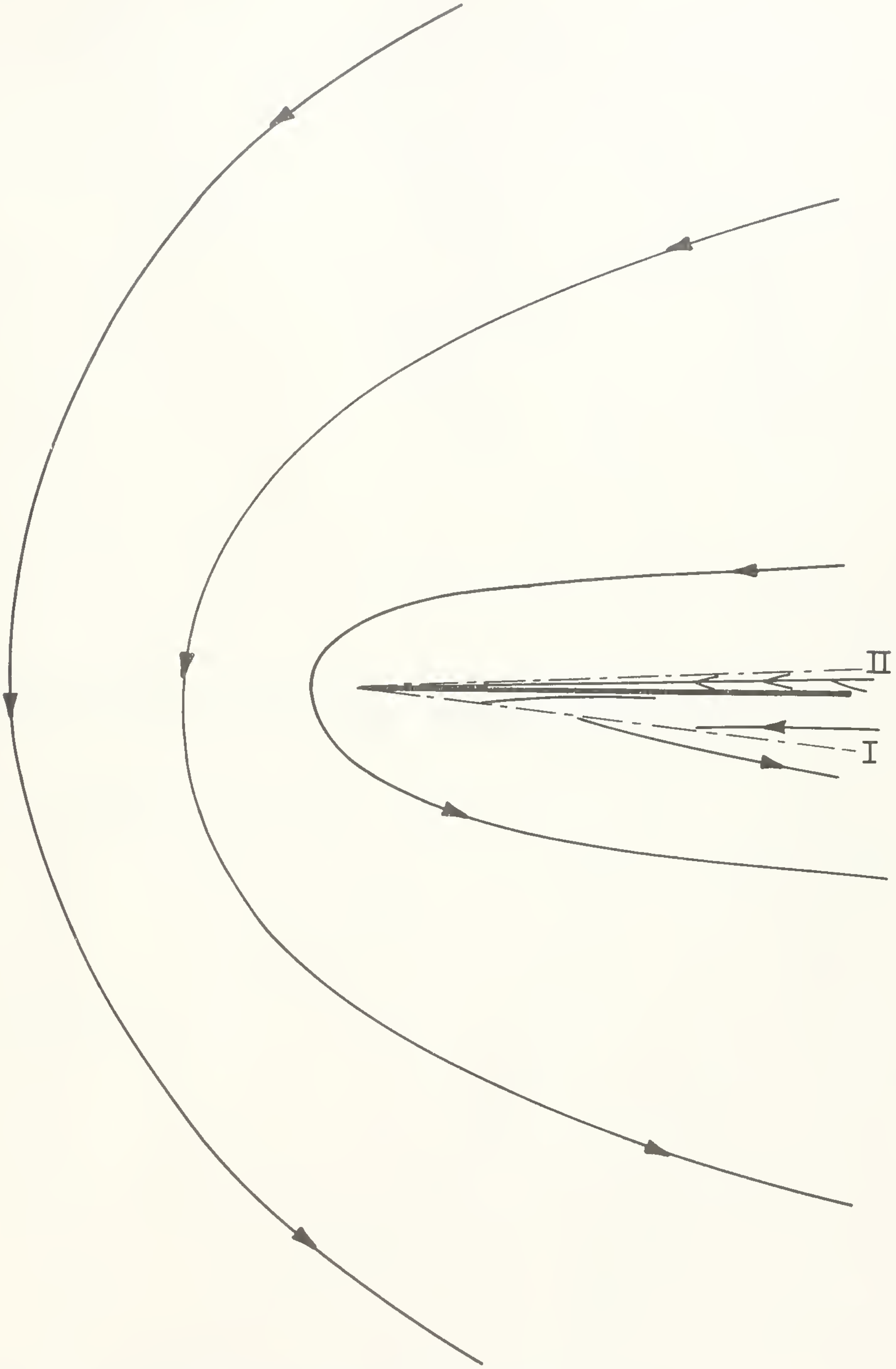

Figure 20: $\sin\theta = .15$ $M_0 = 2.2$ $\alpha = 10.$
LEADING EDGE FLOW $1 - m = .53228$
FLOW SUPPRESSED

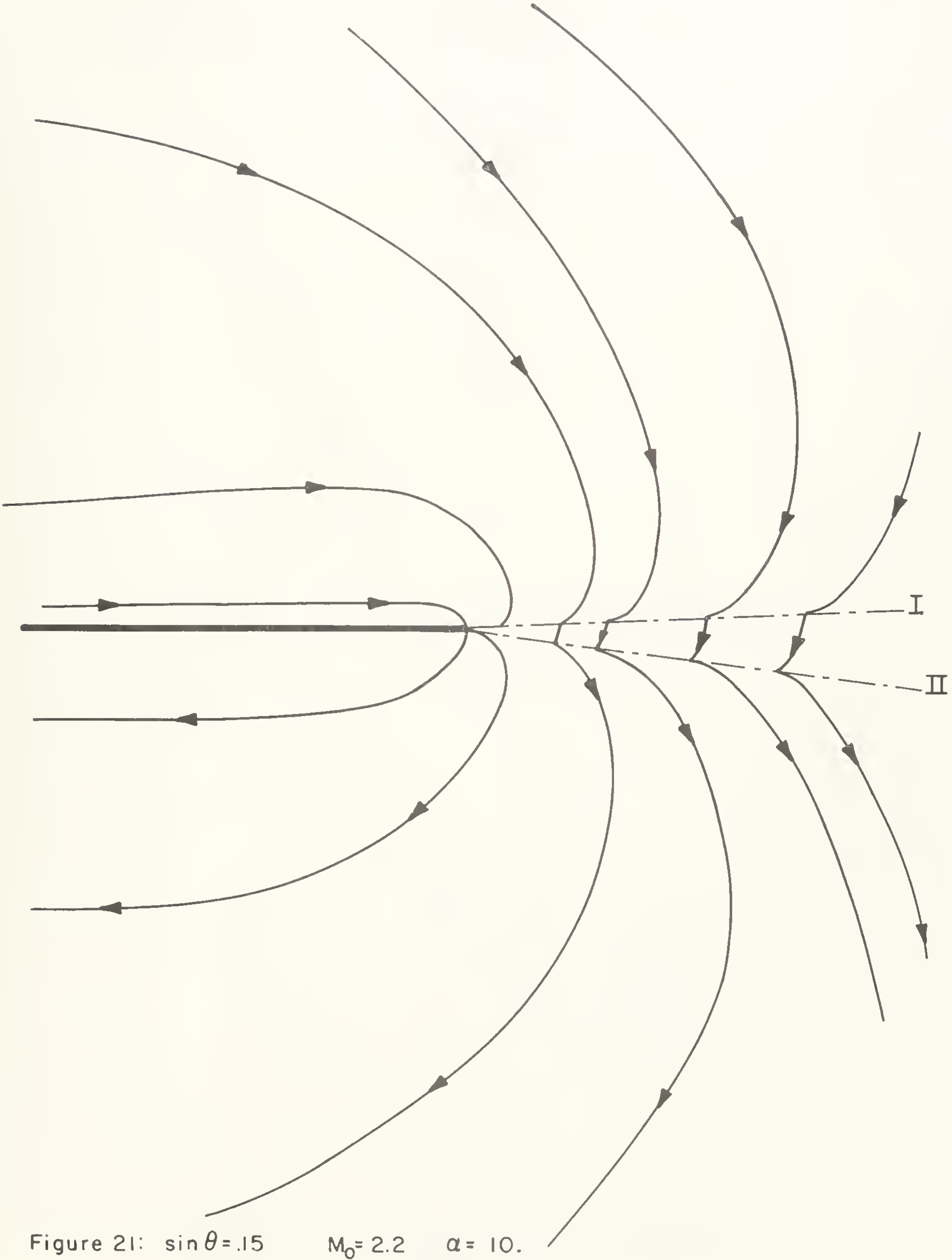

Figure 21: $\sin\theta = .15$ $M_0 = 2.2$ $\alpha = 10.$

TRAILING EDGE FLOW $m = .46772$

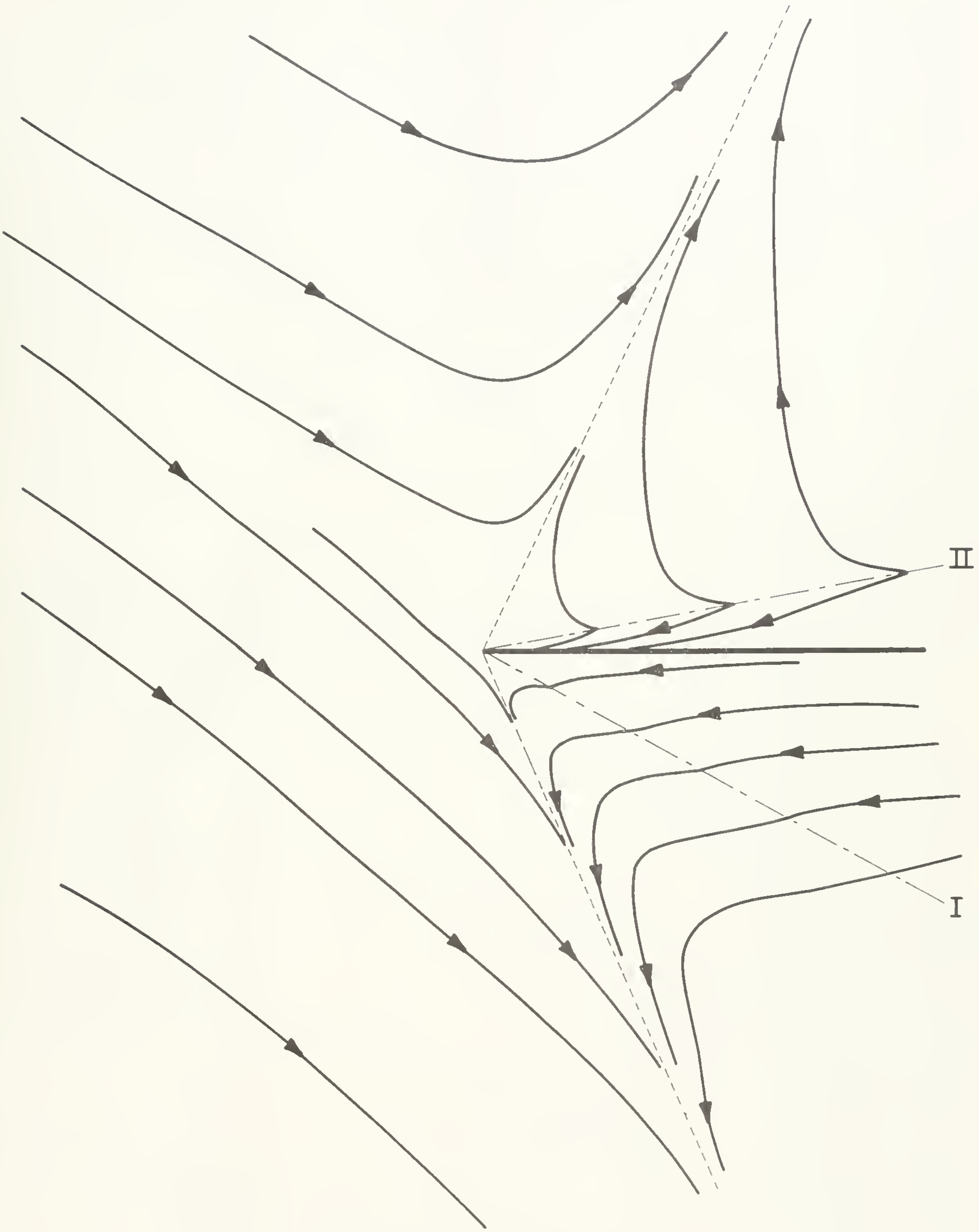

Figure 22: $\sin\theta = .45$ $M_0 = .95$ $\alpha = .5$
LEADING EDGE FLOW $1-m = .62326$
FLOW SUPPRESSED

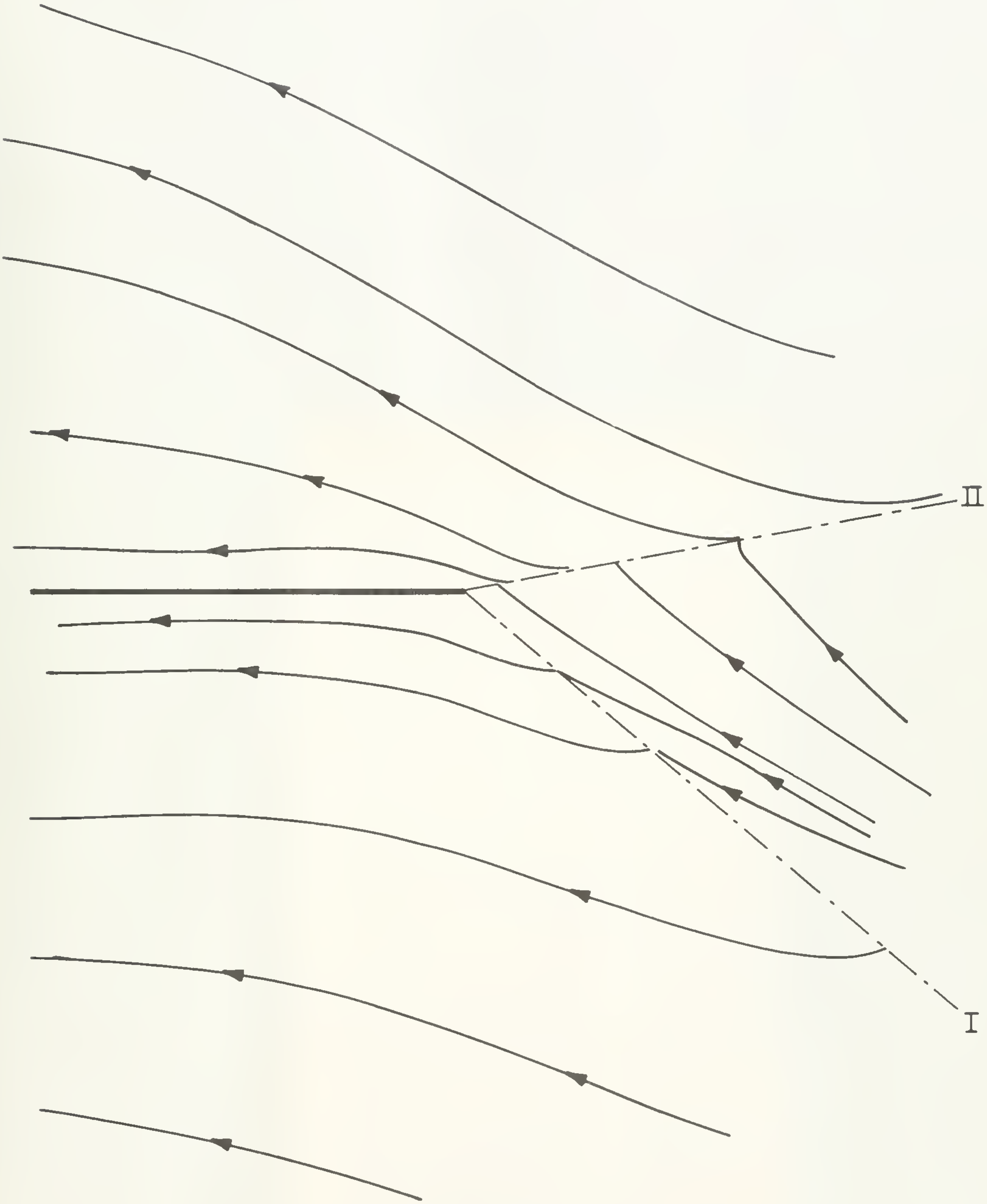

Figure 23: $\sin\theta = .45$ $M_0 = .95$ $\alpha = .5$
TRAILING EDGE FLOW $m = .37674$

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